

A JUMP-TYPE SDE APPROACH TO REAL-VALUED SELF-SIMILAR MARKOV PROCESSES

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ABSTRACT. In his 1972 paper, John Lamperti characterized all positive self-similar Markov processes as time-changes of exponentials of Lévy processes. In the past decade the problem of classifying all non-negative self-similar Markov processes that do not necessarily have zero as a trap has been solved gradually via connections to ladder height processes and excursion theory. Motivated by the recent article [13], we classify via jump-type SDEs the symmetric real-valued self-similar Markov processes that only decrease the absolute value by jumps and leave zero continuously.

Our construction of these self-similar processes involves a pseudo excursion construction and singular stochastic calculus arguments ensuring that solutions to the SDEs spend zero time at zero to avoid problems caused by a "bang-bang" drift.

1. INTRODUCTION AND MAIN RESULTS

1.1. The Classification Problem for Self-Similar Markov Processes. Dating back to Lamperti's seminal article [27], the study of self-similar Markov processes (originally called semi-stable processes by Lamperti) with values in a subset E of \mathbb{R} has attracted a lot of attention. In what follows, we will only discuss self-similar Markov processes in $E = \mathbb{R}$ and $E = [0, \infty)$ and denote by \mathbb{D} the space of càdlàg functions $\omega : \mathbb{R}_+ \rightarrow E$ (right continuous with left limits) endowed with the Borel sigma-field \mathcal{D} generated by Skorokhod's topology. A strong Markov family $(P^z)_{z \in E}$ on $(\mathbb{D}, \mathcal{D})$ is called self-similar if the coordinate process $Z_t(\omega) := \omega(t), t \geq 0$, fulfills the following scaling property:

$$\text{the law of } (c^{-1}Z_{ct})_{t \geq 0} \text{ under } P^z \text{ is } P^{c^{-1}z} \quad (1.1)$$

for all $c > 0$ and $z \in E$. Typically, a more general self-similarity definition is given, replacing the power -1 by $-a$ for some $a > 0$. In this treatment we fix without loss of generality the index of self-similarity $a = 1$ since the change between index a and 1 can be performed by taking the power $Z^{1/a}$. We will say that Z (or alternatively the law $(P^z)_{z \in E}$) is a

- positive self-similar Markov process if $E = [0, \infty)$ and Z is trapped at zero,
- non-negative self-similar Markov process if $E = [0, \infty)$,
- $\mathbb{R} \setminus \{0\} =: \mathbb{R}_*$ -valued self-similar Markov process if $E = \mathbb{R}$ and Z is trapped at zero,
- real-valued self-similar Markov process if $E = \mathbb{R}$.

According to this definition, a positive self-similar Markov process is not really a positive process but the above classification seems to be the most rigorous to separate the appearing cases. Note that, if $T_0 = \inf\{t \geq 0 : Z_t = 0\}$ denotes the first hitting time of zero and $(Z_t^\dagger)_{t \geq 0} := (Z_{t \wedge T_0})_{t \geq 0}$ the process obtained from Z by absorption at 0, then in our notation a non-negative self-similar Markov process contains a positive self-similar Markov process and analogously a real- contains an \mathbb{R}_* -valued self-similar Markov process.

We are interested in the following problem:

Problem. *Classify all positive, non-negative, \mathbb{R}_* - and real-valued self-similar Markov processes.*

The first three instances of the problem have been resolved and only the last remains open.

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(I) *Lamperti's Classification of Positive Self-Similar Markov Processes.* The fundamental result in the classification theory of self-similar Markov processes is Lamperti's representation obtained in [27]. Lamperti showed that there is a bijection between positive self-similar Markov processes and Lévy processes, possibly killed at an independent exponential time ζ . For a Lévy process ξ , Lamperti's representation of positive self-similar Markov processes takes the form

$$Z_t = z \exp(\xi_{\tau(tz^{-1})}), \quad 0 \leq t < T_0, \quad (1.2)$$

where the random time-change is given by the generalized inverse of the exponential functional of ξ , that is

$$\tau(t) := \inf \left\{ s \geq 0 : \int_0^s \exp(\xi_r) dr > t \right\}.$$

It is important to add that T_0 is finite almost surely for all initial conditions $z > 0$ precisely if ξ drifts to $-\infty$ and in this case $\tau(T_0 z^{-1}) = \infty$. Consequently, if we suppose that ξ is set to $-\infty$ at the killing time ζ , then Equation (1.2) is equally valid for $t \geq 0$ with $Z_t = 0$ for any $t \geq T_0$. A consequence of Lamperti's representation is that the Feller property holds on $(0, \infty)$ for any non-negative self-similar strong Markov process.

Example 1.1. *The only positive self-similar Markov processes (with self-similarity index $a = 1$) with continuous sample paths are solutions to the stochastic differential equations (SDEs)*

$$Z_t = a dt + \sigma \sqrt{Z_t} dB_t, \quad t \leq T_0, \quad (1.3)$$

for $a \in \mathbb{R}, \sigma \geq 0$. Their Lamperti transformed Lévy processes are $\xi_t = (a - \frac{\sigma^2}{2})t + \sigma B_t$.

Lamperti's representation was successfully applied for the study of stable Lévy processes since it allows via the identity $T_0 \stackrel{\mathcal{L}}{=} \int_0^\infty \exp(\xi_r) dr$ the study of the exponential functional of a Lévy process via the first hitting time of a self-similar process. For the use of Lamperti's transformation for the study of the maximum of stable Lévy processes we refer for instance to Patie [29] or Kutznetsov and Pardo [26] and references therein.

(II) *Classification of Non-Negative Self-Similar Markov Processes.* We mentioned above that a non-negative self-similar Markov process Z contains a unique positive self-similar Markov process Z^\dagger by absorption at zero. As a consequence, the classification problem for non-negative self-similar Markov processes is equivalent to finding all self-similar extensions of positive self-similar Markov processes. The task has been resolved in recent years; first, if the corresponding Lévy process ξ drifts to $-\infty$ (i.e. $T_0 < \infty$ a.s.), and later if ξ fluctuates or drifts to $+\infty$ (i.e. $T_0 = \infty$ a.s.).

It has first been proved independently by Rivero [32], [33] and Fitzsimmons [19] that positive self-similar Markov processes that hit zero in finite time can be extended uniquely to a non-negative self-similar Markov process that leaves zero continuously if and only if the Cramér type condition

$$\text{there is a } 0 < \theta < 1 \text{ such that } \Psi(\theta) = 0 \quad (1.4)$$

holds. Here, and in what follows, whenever well-defined

$$\Psi(\theta) = \log E(e^{\theta \xi_1}; \zeta > 1), \quad \theta \geq 0,$$

denotes the Laplace exponent of the Lévy process ξ (killed at ζ) that occurs in Lamperti's representation. The proofs of Rivero and Fitzsimmons are based on Blumenthal's general theory of Markov extensions developed in [10].

For positive self-similar Markov processes not hitting zero in finite time, the classification problem is to give conditions when (and how) the family $(P^z)_{z>0}$ can be extended continuously to $z = 0$ so that the extended process remains self-similar and leaves zero. This challenging question was answered subsequently in Bertoin and Caballero [5], Bertoin and Yor [7], Caballero and Chaumont [12] and Chaumont et al. [14]: The law $(P^z)_{z>0}$ of a positive self-similar Markov process extends continuously to initial condition $z = 0$ if and only if

$$\text{the overshoot process } (\xi_{T_x} - x)_{x \geq 0} \text{ converges weakly as } x \rightarrow \infty. \quad (1.5)$$

If the overshoot process does not converge, then the laws P^z converge, as $z \rightarrow 0$, to the degenerate law concentrated at $+\infty$.

The necessity of Condition (1.5) is relatively easy to prove and the main difficulty lies in the construction of P^0 if Condition (1.5) is valid. For the simplest construction of the non-degenerate limiting law P^0 via Lévy processes started from $-\infty$, we refer to Bertoin and Savov [6]. A jump-type SDE approach which motivated the present article was developed in [17].

1.2. (III) Classification of \mathbb{R}_* -Valued Self-Similar Markov Processes. In contrast to self-similar Markov processes with non-negative sample paths, less is known about the classification of self-similar Markov processes with real-valued sample paths. The analogue to Lamperti's representation, called Lamperti-Kiu representation, has recently been proved in Chaumont et al. [13] completing earlier work of Kiu [24], [25] and Chybiryakov [15]. To the best knowledge of the author the classification of real-valued self-similar Markov processes that leave zero remains open.

The main idea of the Lamperti-Kiu representation is as follows: due to the assumed càdlàg property of sample paths, the times H_n of the n -th change of sign

$$H_0 = 0, \quad H_n = \inf\{t > H_{n-1} : Z_t Z_{t-} < 0\}, \quad n \geq 1, \quad (1.6)$$

can only accumulate at T_0 . In the random intervals $[H_n, H_{n+1})$ the real-valued self-similar Markov process reduces to a strictly positive or strictly negative self-similar Markov process to which Lamperti's transformation can be applied and leads to two (possibly different) sequences $\xi^{+,n}$ and $\xi^{-,n}$ of Lévy processes. Using the strong Markov property of Z , independence of the sequence $\xi^{\pm,n}$ follows so that the Lamperti-Kiu representation is obtained by glueing a sequence of Lamperti representations. A crucial additional ingredient are jumps ΔZ_{H_n} that determine the random initial condition for the positive/negative self-similar Markov processes on $[H_n, H_{n+1})$. Again by the strong Markov property it was shown that those jumps are independent and the rate of their occurrence is determined by a random time-change as in Lamperti's representation (1.2). Loosely speaking, the time-change accelerates all jumps with a rate $1/|Z_{s-}|$ and, consequently, Z changes sign infinitely often before and after touching zero. The jumps ΔZ_{H_n} add many difficulties to the classification and prevent a straight forward adaption of arguments developed for positive self-similar Markov processes.

Our main results are for symmetric real-valued self-similar Markov processes, that is the law of $-Z$ under P^z is P^{-z} . In the symmetric case the Lamperti-Kiu transformed Lévy processes satisfy $\xi^{+,n} \stackrel{\mathcal{L}}{=} \xi^{-,n}$.

A formal description of the Lamperti-Kiu representation is rather unpleasant since the change of sign is coded in the underlying Lévy process via an additional complex direction. We follow a different approach based on jump-type SDEs that seems to be more tracktable.

Notations. *Solutions to SDEs are always considered on a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ that is rich enough to carry all appearing Brownian motions and Poisson point processes and satisfies the usual conditions. All SDEs are driven by a (\mathcal{G}_t) -standard Wiener process B and an independent (\mathcal{G}_t) -Poisson random measure \mathcal{N} . We will use weak solutions, i.e. $(\mathcal{G}_t)_{t \geq 0}$ adapted stochastic processes $(Z_t)_{t \geq 0}$ with almost surely càdlàg sample paths that satisfy an SDE in integrated form almost surely. If additionally Z is adapted to the augmented filtration generated by B and \mathcal{N} , then Z is said to be a strong solution. Pathwise uniqueness holds if for any two weak solutions defined on the same probability space with the same standard Wiener process and Poisson random measure, they are indistinguishable. In several SDEs the sign-function*

$$\text{sign}(x) := \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x \leq 0\}}$$

is used. We say a stochastic process Z does not spend time at zero if almost surely

$$\int_0^\infty \mathbf{1}_{\{Z_s = 0\}} ds = 0.$$

Here is a reformulation of the main result of [13] via jump-type SDEs which we only state for symmetric \mathbb{R}_* -valued self-similar Markov processes with the additional assumption

$$(A) \quad P^z(|Z_s| \leq |Z_{s-}|, \forall s \geq 0) = 1, \quad z \in \mathbb{R}_*.$$

Assumption (A) excludes the possibility that jumps of Z increase the absolute value or, equivalently, that the Lévy processes $\xi^{+,n} \stackrel{\mathcal{L}}{=} \xi^{-,n}$ of the Lamperti-Kiu representation have positive jumps. A general non-symmetric version without Assumption (A) is given below in Proposition 2.1.

Proposition 1.2. (I) *There is a bijection between symmetric \mathbb{R}_* -valued self-similar Markov processes satisfying Assumption (A) and quintuples (a, σ^2, Π, q, V) consisting of*

- *a triplet (a, σ^2, Π) of a spectrally negative Lévy process killed at rate q with Laplace exponent Ψ ,*
- *a finite measure $V(du)$ on $[-1, 0)$.*

(II) *For a quintuple (a, σ^2, Π, q, V) a symmetric real-valued self-similar Markov processes issued from $z \in \mathbb{R}_*$ can be constructed as strong solution to*

$$\begin{aligned} Z_t = z + & \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}(Z_s) ds + \sigma \int_0^t \sqrt{|Z_s|} dB_s \\ & + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^1 Z_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned} \quad (1.7)$$

for $t \leq T_0$. Here, B is a standard Brownian motion and \mathcal{N} is a Poisson point process on $(0, \infty) \times (0, \infty) \times [-1, 1]$ with intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \bar{\Pi}(du)$ according to the piecewise definition

- $\bar{\Pi}|_{(0,1]}(du)$ is the image measure under $\mathbb{R}_- \ni u \mapsto e^u$ of Π ,
- $\bar{\Pi}(\{0\}) = q$,
- $\bar{\Pi}|_{[-1,0)}(du) = V(du)$.

Let us briefly explain the ingredients of the jump-type SDE (1.7). Comparing with Example 1.1 the so-called "bang-bang" drift and the Brownian part might be not surprising since in the intervals $[H_n, H_{n+1})$, the restrictions of Z (resp. $-Z$) have to be positive self-similar Markov processes. The jumps of the Poissonian integral are such that

$$Z_{s-} \mapsto Z_{s-} + Z_{s-}(u-1) = Z_{s-}u, \quad (1.8)$$

and u is chosen according to the measure $\bar{\Pi}$ which looks a bit complicated. We chose this formulation since it allows us to explain the three occurring jump possibilities for self-similar Markov processes with only one stochastic integral:

- If $u > 0$, then Z does not change sign and consequently these are the jumps corresponding to a piecewise Lamperti transformation in $[H_n, H_{n+1})$. If the Lévy measure Π is infinite, then also $\bar{\Pi}$ is infinite with a possible pole only at $+1$ so that small jumps (i.e. $\Delta Z_s \approx 0$) accumulate.
- If $u = 0$, then Z jumps to zero which is equivalent to a jump to $-\infty$ (killing) for the Lévy process in Lamperti's representation (1.2).
- If $u < 0$, then Z changes sign and the jump-times are precisely the H_n from (1.6). The finiteness of the intensity measure $V(du)$ is equivalent to the non-accumulation of H_n away from T_0 .

The dr -integral is included to dynamically accelerated the jump rate by $1/|Z_{s-}|$. Hence, on the zero set of solutions all jumps come with infinite rate and the jumps not changing sign even with "double-infinite" rate if Π is infinite. Such explosions of the jump rate are the main difficulty of the SDE (1.7) when studied for all $t \geq 0$ or issued from $z = 0$.

Definition 1.3. (a) For a symmetric \mathbb{R}_* -valued self-similar Markov process the quintuple (a, σ^2, Π, q, V) appearing in Proposition 1.2 (or appearing equivalently in the Lamperti-Kiu representation of [13]) is called the corresponding Lamperti-Kiu quintuple.

(b) A quintuple (a, σ^2, Π, q, V) is called the Lamperti-Kiu quintuple of a symmetric real-valued self-similar Markov process if it is the Lamperti-Kiu quintuple for the \mathbb{R}_* -valued self-similar Markov process obtained by absorption at zero.

1.3. Main Result. The striking feature of the SDE (1.7) compared to the time-change Lamperti-Kiu representation is that the form of possible extensions after hitting zero can be guessed immediately. If possible, they should be solutions to the same SDE for all $t \geq 0$. Here is the main result of this article:

Theorem 1.4. (I) There is a bijection between symmetric real-valued self-similar Markov processes that leave zero continuously and satisfy Assumption (A) and quintuples (a, σ^2, Π, q, V) consisting of

- a triplet (a, σ^2, Π) of a spectrally negative Lévy process killed at rate q with Laplace exponent Ψ ,
- a finite measure $V(du)$ on $[-1, 0)$

that satisfy

$$\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) > 0. \quad (1.9)$$

(II) For a quintuple (a, σ^2, Π, q, V) as in (I) a symmetric real-valued self-similar Markov processes that leaves zero continuously and is issued from $z \in \mathbb{R}$ can be constructed as weak solution to the SDE (1.7) for $t \geq 0$.

The necessity of Condition (1.9) can be found easily by a reduction to Condition (1.4) for positive self-similar Markov processes. The difficult part of the proof is a construction of a real-valued self-similar Markov process with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) whenever Condition (1.9) is valid. Our reformulation of the Lamperti-Kiu representation given in Proposition 1.2 turns out to be useful since it gives flexibility for the construction via approximation procedures and semimartingale calculus. The approximation is rather non-standard (and might remind the reader to constructions of skew Brownian motion) since the non-continuity of the "bang-bang" drift causes problems in weak convergence arguments. Limiting points of the approximating sequences might become trivial (trapped at zero) and it is precisely Condition (1.9) that ensures this is not the case.

Remark 1.5. It is surprising that Condition (1.9) is sufficient for the existence of solutions to the SDE (1.7) that leave zero. Since Ψ does not depend on V , the quintuple (a, σ^2, Π, q, V) can be chosen such that

$$\left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) < 0 < \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right).$$

Then the SDE (1.7) has martingale terms that vanish at zero and a drift that points towards the origin. In such a situation it is impossible to find non-negative solutions to SDEs since positive martingales are absorbed at zero. Real-valued solutions, however, can exist. Precisely the jumps crossing the origin cause this effect; if $V = 0$ solutions can leave if and only if the drift points away from the origin.

It is important to note that the SDE (1.7) behaves very differently at zero and away from zero. Away from zero the coefficients are locally Lipschitz continuous so that pathwise uniqueness holds and strong solutions exist. Only when solutions touch zero the drift and the jumps are problematic. Consequently, the main task of the proofs is to give a construction and uniqueness statement for solutions issued from zero.

1.4. Connection to Other Self-Similar SDEs. Theorem 1.4 extends the results of [17] obtained for positive self-similar Markov processes with an assumption similar to Assumption **(A)**. The techniques utilized here need to be different from those of [17] since the drift coefficient is discontinuous so that standard arguments for SDEs do not apply. In particular, solutions to (1.7) are constructed even if the drift points towards zero which forces us to leave classical arguments in the spirit of Yamada and Watanabe and combine more specific stochastic calculus arguments with general martingale problem techniques. The main result of [17] was stronger in the sense that pathwise uniqueness could be proved for their SDEs and solutions are automatically strong. Consequently, the constructed non-negative self-similar Markov processes are deterministic functionals of a Brownian motion and a Poisson point process so that we can speak of a strong classification.

Remark 1.6. *Possible uniqueness statements for the SDE (1.7) issued from $z = 0$ need to be in a restricted sense as one can see at the simplest special case*

$$dZ_t = \text{sign}(Z_t) dt + 2\sqrt{|Z_t|} dB_t, \quad Z_0 = 0. \quad (1.10)$$

Given a Brownian motion W , then three weak solution to the SDE (1.10) can be defined explicitly:

$$Z_t^{(1)} = \text{sign}(W_t)W_t^2, \quad Z_t^{(2)} = W_t^2, \quad Z_t^{(3)} = -W_t^2.$$

Of course, already the one-dimensional marginal distributions differ for the $Z^{(i)}$. Nonetheless, restricted to symmetric solutions (this rules out $Z^{(2)}$ and $Z^{(3)}$) one can easily deduce the uniqueness for the one-dimensional marginals. Tanaka's formula applied to (1.10) shows that the absolute value of any solution satisfies

$$X_t = t + 2 \int_0^t \sqrt{X_s} dB_s. \quad (1.11)$$

Now uniqueness for (1.11) implies uniqueness for the absolute value of solutions to (1.10), hence, uniqueness for one-dimensional marginals of symmetric solutions.

The simple example shows that the best possible uniqueness statement for solutions to the SDE (1.7) is pathwise uniqueness among symmetric solutions.

There might be more sophisticated arguments that yield pathwise uniqueness among symmetric solutions, such as the arguments developed in Bass et al. [11] for the self-similar SDE

$$dZ_t = |Z_t|^\beta dB_t, \quad t \geq 0, \quad (1.12)$$

for $\beta < 1/2$. They work under the restriction to solutions that do not spend time at zero, a property which is also crucial in all our arguments. Note that the index of self-similarity of (1.12) is $a = \frac{1}{2-2\beta} < 1$ and the Hölder continuity of the coefficient becomes worse when the self-similarity index decreases. For the classification problem of real-valued self-similar Markov processes the assumption $a = 1$ could be imposed without loss of generality but it would be interesting to see if pathwise uniqueness among symmetric solutions holds for the generalized version of the SDE (1.7) that should describe all symmetric real-valued self-similar Markov processes of index a :

$$\begin{aligned} Z_t = z + \sigma \int_0^t |Z_s|^{1-\frac{1}{2a}} dB_s + \left(\Psi(a) + \int_{-\infty}^0 (u-1)V(du) \right) \int_0^t \text{sign}(Z_s) Z_s^{1-\frac{1}{a}} ds \\ + \int_0^t \int_0^{\frac{1}{|Z_{s-}|^a}} \int_{\mathbb{R}} Z_{s-} (u-1) (\mathcal{N} - \mathcal{N}') (ds, dr, du), \end{aligned}$$

with the same definitions as in Proposition 1.2. This generalization of (1.7) can be derived from the Lamperti-Kiu representation as we do in Section 2.1 for the special case $a = 1$. For the drift and diffusive coefficients the self-similarity index $a = 1$ separates between a regime of Hölder continuity ($a > 1$) and a regime with singular drift ($a < 1$). Moreover, for all $a > 0$ we find lack of monotonicity in the Poissonian integral and it seems that this terms forces the biggest troubles.

Organization of the Article. In Section 2.1 we prove the jump-type SDE reformulation of the Lamperti-Kiu representation for real-valued self-similar Markov processes. The proofs are given for the more general setup without symmetry and without the Assumption **(A)**. The construction of solutions to (1.7) that leave zero is presented in Section 2.2. Self-similarity and the strong Markov property are deduced from moment equations which imply uniqueness of one-dimensional marginals for solutions to the SDE (1.7). Finally, the link to the classification problem of self-similar Markov processes is given in Section 2.3

2. PROOFS

2.1. Lamperti-Kiu Representation via Jump-Type SDEs. We now state and prove a jump-type SDE formulation of the Lamperti-Kiu representation in the general case.

Proposition 2.1. *(I) There is a bijection between \mathbb{R}_* -valued self-similar Markov processes and two quintuples $(a_+, \sigma_+^2, \Pi_+, q_+, V_+)$ and $(a_-, \sigma_-^2, \Pi_-, q_-, V_-)$ consisting of*

- *two triplets $(a_\pm, \sigma_\pm^2, \Pi_\pm)$ of Lévy processes killed at rates q_\pm with Laplace exponents Ψ_\pm ,*
- *two finite measures $V_\pm(du)$ on $(-\infty, 0)$.*

(II) Given two quintuples $(a_\pm, \sigma_\pm^2, \Pi_\pm, q_\pm, V_\pm)$, a real-valued self-similar Markov processes issued from $z \in \mathbb{R}_$ can be constructed as strong solution to*

$$\begin{aligned} Z_t = z + & \left[\left(a_+ + \frac{\sigma_+^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi_+(du) \right) \int_0^t \mathbf{1}_{\{Z_s > 0\}} ds + \sigma_+ \int_0^t \sqrt{|Z_s|} \mathbf{1}_{\{Z_s > 0\}} dB_+(s) \right. \\ & + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{\mathbb{R} \setminus (1/e, e)} \mathbf{1}_{\{Z_{s-} > 0\}} Z_{s-} (u - 1) \mathcal{N}_+(ds, dr, du) \\ & + \left. \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{1/e}^e \mathbf{1}_{\{Z_{s-} > 0\}} Z_{s-} (u - 1) (\mathcal{N}_+ - \mathcal{N}_+')(ds, dr, du) \right] \\ & + \left[\left(a_- + \frac{\sigma_-^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi_-(du) \right) \int_0^t \mathbf{1}_{\{Z_s < 0\}} ds + \sigma_- \int_0^t \sqrt{|Z_s|} \mathbf{1}_{\{Z_s < 0\}} dB_-(s) \right. \\ & + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{\mathbb{R} \setminus (1/e, e)} \mathbf{1}_{\{Z_{s-} < 0\}} Z_{s-} (u - 1) \mathcal{N}_-(ds, dr, du) \\ & + \left. \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{1/e}^e \mathbf{1}_{\{Z_{s-} < 0\}} Z_{s-} (u - 1) (\mathcal{N}_- - \mathcal{N}_-')(ds, dr, du) \right], \quad t \leq T_0. \end{aligned}$$

Here, B_\pm are standard Brownian motions and \mathcal{N}_\pm are independent Poisson point processes on $(0, \infty) \times (0, \infty) \times (-\infty, \infty)$ with intensity measure $\mathcal{N}'_\pm(ds, dr, du) = ds \otimes dr \otimes \bar{\Pi}_\pm(du)$ according to the piecewise definition

- $\bar{\Pi}_\pm^\pm(du)$ are the image measures under $\mathbb{R} \ni u \mapsto e^u$ of Π_\pm ,
- $\bar{\Pi}_\pm^\pm(\{0\}) = q_\pm$,
- $\bar{\Pi}_\pm^\pm(du) = V_\pm(du)$.

Before proving the proposition, let us quickly consider part (II) for two special cases.

Example 2.2. *With the choice $z > 0$ and*

$$\begin{aligned} (a_+, \sigma_+^2, \Pi_+, q_+, V_+) &= (a, \sigma^2, \Pi, q, 0) \\ (a_-, \sigma_-^2, \Pi_-, q_-, V_-) &= (0, 0, 0, 0, 0) \end{aligned} \tag{2.1}$$

zero is not crossed and, dropping the subscripts, the SDE simplifies to

$$\begin{aligned} Z_t = z &+ \left(a + \frac{\sigma^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi(du) \right) t + \sigma \int_0^t \sqrt{Z_s} dB(s) \\ &+ \int_0^t \int_0^{\frac{1}{Z_{s-}}} \int_{\mathbb{R}_+ \setminus (e^{-1}, e)} Z_{s-} (u - 1) \mathcal{N}(ds, dr, du) \\ &+ \int_0^t \int_0^{\frac{1}{Z_{s-}}} \int_{e^{-1}}^e Z_{s-} (u - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0. \end{aligned}$$

Under the additional assumption $\int_{\mathbb{R}_+ \setminus (e^{-1}, e)} (u - 1) \bar{\Pi}(du) = \int_{|u| > 1} (e^u - 1) \Pi(du) < \infty$ one can use the Lévy-Khintchin representation to simplify by adding and subtracting the finite compensator integral to get

$$\begin{aligned} Z_t = z &+ \Psi(1)t + \sigma \int_0^t \sqrt{Z_s} dB_s \\ &+ \int_0^t \int_0^{\frac{1}{Z_{s-}}} \int_0^\infty Z_{s-} (u - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0. \end{aligned} \tag{2.2}$$

The SDE (2.2) was already derived in [17] as reformulation of Lamperti's transformation for positive self-similar Markov processes; strong existence and pathwise uniqueness for $t \geq 0$ was proved in [17] and also by Li and Pu [28].

The next example shows that Proposition 1.2 is a special case of Proposition 2.1 noting that (2.3) holds trivially if $\bar{\Pi}$ is concentrated on $[-1, 1]$.

Example 2.3. *Let us assume the symmetry*

$$(a_+, \sigma_+^2, \Pi_+, q_+, V_+) = (a_-, \sigma_-^2, \Pi_-, q_-, V_-)$$

and

$$\int_{|u| > 1} (e^u - 1) \Pi_\pm(du) + \int_{-\infty}^0 u V_\pm(du) < \infty. \tag{2.3}$$

The noises B_\pm, \mathcal{N}_\pm can be replaced in this special case by B, \mathcal{N} due to the symmetry assumption and the independence of increments. Adding and subtracting the compensator integral as in Example 2.2 yields the SDE (1.7).

Proof of Proposition 2.1. To save notation, let us assume throughout the proof $z > 0$; for $z < 0$ the arguments follow the same lines interchanging odd and even.

We start with a reminder of the main result of [13]: the Lamperti-Kiu representation formulated as time-changed exponential of a complex-valued Lévy process. Let ξ^\pm be real-valued Lévy processes with triplets $(a_\pm, \sigma_\pm^2, \Pi_\pm)$ killed at rates q_\pm (formalized here as jump to $-\infty$ without causing technical complication since the process will be absorbed at the first occurrence) as in the formulation of the proposition, ζ^\pm exponential random variables with parameters $p_\pm = V_\pm(\mathbb{R})$. If we denote (without confusion) by V_\pm equally negative random variables with probability distribution $V_\pm(du)/V_\pm(\mathbb{R})$, then $U^\pm := \log(|V_\pm|)$ are real-valued random-variables (for the trivial case $V_\pm(\mathbb{R}) = 0$ we define $U^\pm = 0$). Further, suppose that $\xi^\pm, \zeta^\pm, U^\pm$ are independent. We consider the sequence $((\xi^k, \zeta^k, U^k), k \geq 0)$ given by

$$(\xi^k, \zeta^k, U^k) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}) : & k \text{ even (including } k = 0), \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}) : & k \text{ odd,} \end{cases} \tag{2.4}$$

where $(\xi^{\pm,k}, \zeta^{\pm,k}, U^{\pm,k}) \stackrel{\mathcal{L}}{=} (\xi^{\pm}, \zeta^{\pm}, U^{\pm})$ are independent. Let $(T_k, k \geq 0)$ be the sequence defined by

$$T_0 = 0, \quad T_n = \sum_{k=0}^{n-1} \zeta^k, \quad n \geq 1,$$

and $(N_t, t \geq 0)$ the alternating renewal type process

$$N_t = \max \{n \geq 0 : T_n \leq t\}.$$

For simplicity, the abbreviation

$$\sigma_t = t - T_{N_t}, \quad \xi_{\sigma_t} = \xi_{\sigma_t}^{N_t}, \quad \xi_{\zeta}^k = \xi_{\zeta^k}^k,$$

is used. With the notation, the Lamperti-Kiu representation becomes

$$Z_t = z \exp(\mathcal{E}_{\tau(t|z|^{-1})}), \quad t \leq T_0, \quad (2.5)$$

where

$$\mathcal{E}_t = \xi_{\sigma_t} + \sum_{k=1}^{N_t-1} (\xi_{\zeta}^k + U^k) + i\pi N_t$$

and

$$\tau(t) := \inf \left\{ s \geq 0 : \int_0^s |\exp(\mathcal{E}_r)| dr > t \right\}.$$

Theorem 6 of [13] states that (2.5) defines an \mathbb{R}_* -valued self-similar Markov process issued from z and conversely every \mathbb{R}_* -valued self-similar Markov process can be represented via (2.5) with two quintuples as in the statement of part (I) of the proposition. Recall that throughout this article we suppose the index of self-similarity is 1.

Note that, if $U^{\pm} = 0$ then $N_t = 0, \sigma_t = t$ and (2.5) simplifies to Lamperti's representation (1.2).

The rest of the proof is concerned with part (II), the reformulation of (2.5) via jump-type SDEs for which the Lévy-Itô representation is applied to the occurring Lévy process $\xi^{\pm,k}$. Since the sequence of Lévy processes is independent and runs on disjoint time-intervals the same driving noises can be used for all $k \geq 1$. The occurring Brownian motions are denoted by W_{\pm}^1 and the Poisson point processes with intensities $ds \otimes \Pi_{\pm}(du)$ by \mathcal{N}_{\pm}^1 . Recall that the killing is included by atoms at $-\infty$ with weights q_{\pm} . Additionally, since the jumps in imaginary direction and $U^{\pm,k}$ come at same times, they can be added according to Poisson point processes \mathcal{M}_{\pm}^1 on $(0, \infty) \times \mathbb{R}$ with intensity measures $\mathcal{M}_{\pm}^1(ds, dv) = p_{\pm} ds \otimes U^{\pm}(du)$, where $U^{\pm}(du)$ denotes the probability law of U^{\pm} . Since $U^{\pm}(du)$ is a probability measure the jump-rate of \mathcal{M}_{\pm}^1 is p_{\pm} . The Lévy-Itô type representation for

\mathcal{E} can now be written as

$$\begin{aligned} \mathcal{E}_t = & \left[a_+ \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s) \text{ even}\}} ds + \sigma_+ \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s) \text{ even}\}} dW_+^1(s) \right. \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ even}\}} u (\mathcal{N}_+^1 - \mathcal{N}_+^{1'})(ds, du) + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ even}\}} u \mathcal{N}_+^1(ds, du) \\ & \left. + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ even}\}} (u + i\pi) \mathcal{M}_+^1(ds, du) \right] \\ & + \left[a_- \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s) \text{ odd}\}} ds + \sigma_- \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s) \text{ odd}\}} dW_-^1(s) \right. \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ odd}\}} u (\mathcal{N}_-^1 - \mathcal{N}_-^{1'})(ds, du) + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ odd}\}} u \mathcal{N}_-^1(ds, du) \\ & \left. + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-}) \text{ odd}\}} (u + i\pi) \mathcal{M}_-^1(ds, du) \right] \end{aligned}$$

with the convention 0 is even. If now we set $\eta_t = z \exp(\mathcal{E}_t)$, apply Itô's lemma and use

$$\mathbf{1}_{\{Im(\mathcal{E}_s) \text{ even}\}} = \mathbf{1}_{\{\eta_s > 0\}}, \quad \mathbf{1}_{\{Im(\mathcal{E}_s) \text{ odd}\}} = \mathbf{1}_{\{\eta_s < 0\}},$$

then we obtain

$$\begin{aligned} \eta_t = & z + \left[a_+ \int_0^t \eta_s \mathbf{1}_{\{\eta_s > 0\}} ds + \frac{\sigma_+^2}{2} \int_0^t \mathbf{1}_{\{\eta_s > 0\}} \eta_s ds + \sigma_+ \int_0^t \eta_s \mathbf{1}_{\{\eta_s > 0\}} dW_+^1(s) \right. \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{\eta_{s-} > 0\}} \eta_{s-} (e^u - 1) (\mathcal{N}_+^1 - \mathcal{N}_+^{1'})(ds, du) + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{\eta_{s-} > 0\}} \eta_{s-} (e^u - 1) \mathcal{N}_+^1(ds, du) \\ & \left. + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{\eta_{s-} > 0\}} \eta_{s-} (e^u - 1 - u) ds \Pi_+(du) + \int_0^t \int_{\mathbb{R}} \eta_{s-} \mathbf{1}_{\{\eta_{s-} > 0\}} (-e^u - 1) \mathcal{M}_+^1(ds, du) \right] \\ & + \left[a_- \int_0^t \eta_s \mathbf{1}_{\{\eta_s < 0\}} ds + \frac{\sigma_-^2}{2} \int_0^t \mathbf{1}_{\{\eta_s < 0\}} \eta_s ds + \sigma_- \int_0^t \eta_s \mathbf{1}_{\{\eta_s < 0\}} dW_-^1(s) \right. \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{\eta_{s-} < 0\}} \eta_{s-} (e^u - 1) (\mathcal{N}_-^1 - \mathcal{N}_-^{1'})(ds, du) + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{\eta_{s-} < 0\}} \eta_{s-} (e^u - 1) \mathcal{N}_-^1(ds, du) \\ & \left. + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{\eta_{s-} < 0\}} \eta_{s-} (e^u - 1 - u) ds \Pi_-(du) + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{\eta_{s-} < 0\}} \eta_{s-} (-e^u - 1) \mathcal{M}_-^1(ds, du) \right]. \end{aligned}$$

To incorporate the time-change τ we follow closely the arguments of Proposition 3.13 of [17] for the special case (2.1). Since the arguments are almost identical, we refer for the verification of intermediate steps to the careful treatment in [17].

Let us first denote by $(t_n, \Delta_n)_{n \in \mathbb{N}}$ an arbitrary labeling of the pairs associated to jump times and jump sizes of $(\mathcal{E}_{\tau(tz^{-1})})_{t \in [0, T_0]}$ and more precisely by $(t_n, \Delta_n^\pm)_{n \in \mathbb{N}}$ the subset of jumps due to \mathcal{N}^\pm and by $(t_n, \bar{\Delta}_n^\pm)_{n \in \mathbb{N}}$ the subset of jumps due to \mathcal{M}^\pm . We can assume that we are given additionally independent Wiener processes $(\bar{W}_\pm(t))_{t \geq 0}$, Poisson random measures \mathcal{P}_\pm^1 on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ with intensity measure $ds \otimes dr \otimes \Pi(du)$ and Poisson point processes \mathcal{P}_\pm^2 on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ with intensity measure $p_\pm ds \otimes dr \otimes U^\pm(du)$ that generate a filtration $(\mathcal{H}_t)_{t \geq 0}$. Additionally, we choose an independent sequence of random variables $(R_n)_{n \in \mathbb{N}}$ uniformly distributed on $(0, 1)$ such

that R_n is \mathcal{H}_{t_n} -measurable and independent of \mathcal{H}_{t_n-} and define

$$\begin{aligned} W_{\pm}^2(t) &= \int_0^t \text{sign}(Z_s) \sqrt{|Z_s|} dW_{\pm}^1(\tau(s|z|^{-1})) + \int_0^t \mathbf{1}_{\{Z_s=0\}} d\bar{W}_{\pm}(s), \\ \mathcal{N}_{\pm}^2(A_1 \times A_2 \times A_3) &= \sum_{n=1}^{\infty} \mathbf{1}_{\{A_1 \times A_2 \times A_3\}}((t_n, R_n/|Z_{t_n-}|, \Delta_n^{\pm})) \\ &\quad + \int_{A_1} \int_{A_2} \int_{A_3} (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}}) \mathcal{P}_{\pm}^1(ds, dr, du), \\ \mathcal{M}_{\pm}^2(A_1 \times A_2 \times A_3) &= \sum_{n=1}^{\infty} \mathbf{1}_{\{A_1 \times A_2 \times A_3\}}((t_n, R_n/|Z_{t_n-}|, \bar{\Delta}_n^{\pm})) \\ &\quad + \int_{A_1} \int_{A_2} \int_{A_3} (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}}) \mathcal{P}_{\pm}^2(ds, dr, du), \end{aligned}$$

for all $A_1, A_2 \in \mathcal{B}((0, \infty))$ and $A_3 \in \mathcal{B}(\mathbb{R})$. It now follows from Lévy's characterization that the W_{\pm}^2 are \mathcal{H}_t -Brownian motions:

$$\langle W_{\pm}^2(\cdot) \rangle_t = \int_0^t |Z_s| \mathbf{1}_{\{Z_s \neq 0\}} d\tau(s|z|^{-1}) + \int_0^t \mathbf{1}_{\{Z_s=0\}} ds = \int_0^t \mathbf{1}_{\{Z_s \neq 0\}} ds + \int_0^t \mathbf{1}_{\{Z_s=0\}} ds = t.$$

Furthermore, to show that the \mathcal{N}_{\pm}^2 are \mathcal{H}_t -Poisson point processes with intensity measures $dt \otimes dr \otimes \Pi_{\pm}(du)$. Applying Theorems II.1.8 and II.4.8 of [22], we need to verify

$$E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) \mathcal{N}_{\pm}^2(ds, dr, du) \right) = E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) ds dr \Pi_{\pm}(du) \right) \quad (2.6)$$

for every non-negative predictable function H on $\Omega \times (0, \infty) \times (0, \infty) \times \mathbb{R}$. By the definition of \mathcal{N}_{\pm}^2 we can write

$$\begin{aligned} &E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) \mathcal{N}_{\pm}^2(ds, dr, du) \right) \\ &= E \left(\sum_{n=1}^{\infty} H(t_n, R_n/|Z_{t_n-}|, \Delta_n) \right) \\ &\quad + E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}}) \mathcal{P}_{\pm}^1(ds, dr, du) \right). \end{aligned} \quad (2.7)$$

To express the first summand we apply Theorem II.1.8 of Jacod and Shiryaev [22] to the non-negative predictable function

$$\tilde{H}(s, r, u) := H(s, r/|Z_{s-}|, u), \quad s > 0, r > 0, u \in \mathbb{R},$$

and the Poisson random measure on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ defined by

$$\tilde{\mathcal{P}}^1(A_1 \times A_2 \times A_3) := \sum_{n=1}^{\infty} \mathbf{1}_{A_1 \times A_2 \times A_3}((t_n, R_n, \Delta_n)), \quad A_1, A_2 \in \mathcal{B}((0, \infty)), \quad A_3 \in \mathcal{B}(\mathbb{R}),$$

to obtain

$$\begin{aligned} E \left(\sum_{n=1}^{\infty} H(t_n, R_n/|Z_{t_n-}|, \Delta_n) \right) &= E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} \tilde{H}(s, r, u) \tilde{\mathcal{P}}^1(ds, dr, du) \right) \\ &= E \left(\int_0^{T_0} \int_0^1 \int_{\mathbb{R}} H(s, r/|Z_{s-}|, u) d\tau(s|z|^{-1}) dr \Pi(du) \right), \end{aligned}$$

where the second equality holds since, by construction, the compensator measure of $\tilde{\mathcal{P}}^1$ is

$$\mathbf{1}_{(0, T_0)}(s) \mathbf{1}_{(0, 1)}(r) d\tau(s|z|^{-1}) dr \Pi(du).$$

Utilizing a change of variable in the second coordinate of H , we can further simplify the right-hand side to

$$\begin{aligned} & E \left(\int_0^{T_0} \int_0^1 \int_{\mathbb{R}} \frac{1}{|Z_{s-}|} H(s, r/|Z_{s-}|, u) ds dr \Pi(du) \right) \\ &= E \left(\int_0^{T_0} \int_0^{1/|Z_{s-}|} \int_{\mathbb{R}} H(s, r, u) ds dr \Pi(du) \right). \end{aligned}$$

Similarly, applying Theorem II.1.8 of Jacod and Shiryaev [22] to \mathcal{P}_{\pm}^1 , the second summand of the right hand side of (2.7) equals

$$\begin{aligned} & E \left(\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}}) \mathcal{P}_{\pm}^1(ds, dr, du) \right) \\ &= E \left(\int_0^{T_0} \int_{1/|Z_s|}^{\infty} \int_{\mathbb{R}} H(s, r, u) ds dr \Pi(du) \right) + E \left(\int_{T_0}^{\infty} \int_0^{\infty} \int_{\mathbb{R}} H(s, r, u) ds dr \Pi(du) \right). \end{aligned}$$

Adding the right hand sides of the above two equalities, by (2.7), we have (2.6).

Similarly, one can show that the \mathcal{M}_{\pm}^2 are \mathcal{H}_t -Poisson point processes with intensity measures $p_{\pm} dt \otimes dr \otimes U^{\pm}(du)$.

Plugging-in the new Brownian motion we obtain

$$\begin{aligned} \sigma_{\pm} \int_0^{\tau(t|z|^{-1})} \mathbf{1}_{\{\eta_s>0\}} \eta_s dW_{\pm}^2(s) &= \sigma_{\pm} \int_0^t \mathbf{1}_{\{Z_s>0\}} Z_s dW_{\pm}^2(\tau(sz^{-1})) \\ &= \sigma_{\pm} \int_0^t \mathbf{1}_{\{Z_s>0\}} \sqrt{|Z_s|} \operatorname{sign}(Z_s) \sqrt{|Z_s|} dW_{\pm}^2(\tau(sz^{-1})) \\ &= \sigma_{\pm} \int_0^t \mathbf{1}_{\{Z_s>0\}} \sqrt{|Z_s|} dW_{\pm}^1(s), \quad t \leq T_0, \end{aligned}$$

and analogously for the negative part. Comparing one-by-one the jumps of the Poisson point processes we also find, by construction of the new point measures,

$$\begin{aligned} & \int_0^{\tau(t|z|^{-1})} \int_{|u|>1} \mathbf{1}_{\{\eta_{s-}>0\}} \eta_{s-} (e^u - 1) \mathcal{N}_{+}^1(ds, du) \\ &= \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u|>1} \mathbf{1}_{\{Z_{s-}>0\}} Z_{s-} (e^u - 1) \mathcal{N}_{+}^2(ds, dr, du), \quad t \leq T_0, \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\tau(t|z|^{-1})} \int_{|u|\leq 1} \mathbf{1}_{\{\eta_{s-}>0\}} \eta_{s-} (e^u - 1) (\mathcal{N}_{+}^1 - \mathcal{N}_{+}^{1'})(ds, du) \\ &= \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u|\leq 1} \mathbf{1}_{\{Z_{s-}>0\}} Z_{s-} (e^u - 1) (\mathcal{N}_{+}^2 - \mathcal{N}_{+}^{2'})(ds, dr, du), \quad t \leq T_0, \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\tau(t|z|^{-1})} \int_{\mathbb{R}} \mathbf{1}_{\{\eta_{s-}>0\}} \eta_{s-} (-e^u - 1) \mathcal{M}_{+}^1(ds, du) \\ &= \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-}>0\}} Z_{s-} (-e^u - 1) \mathcal{M}_{+}^2(ds, dr, du), \quad t \leq T_0, \end{aligned}$$

and analogously for the negative parts. Finally, ordinary change of time yields

$$\frac{\sigma_{\pm}^2}{2} \int_0^{\tau(t|z|^{-1})} \eta_s \mathbf{1}_{\{\eta_s>0\}} ds = \frac{\sigma_{\pm}^2}{2} \int_0^t \mathbf{1}_{\{Z_s>0\}} ds, \quad t \leq T_0.$$

Plugging-into the integral equation derived for η , we find that Z satisfies

$$\begin{aligned}
Z_t = z + & \left[\left(a_+ + \frac{\sigma_+^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi_+(du) \right) \int_0^t \mathbf{1}_{\{Z_s > 0\}} ds + \sigma_+ \int_0^t \sqrt{|Z_s|} \mathbf{1}_{\{Z_s > 0\}} dW_+^2(s) \right. \\
& + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u| > 1} \mathbf{1}_{\{Z_{s-} > 0\}} Z_{s-} (e^u - 1) \mathcal{N}_+^2(ds, dr, du) \\
& + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u| \leq 1} \mathbf{1}_{\{Z_{s-} > 0\}} Z_{s-} (e^u - 1) (\mathcal{N}_+^2 - \mathcal{N}_+^{2'}) (ds, dr, du) \\
& \left. + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-} > 0\}} Z_{s-} (-e^v - 1) \mathcal{M}_+^2(ds, dr, dv) \right] \\
& + \left[\left(a_- + \frac{\sigma_-^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi_-(du) \right) \int_0^t \mathbf{1}_{\{Z_s < 0\}} ds + \sigma_- \int_0^t \sqrt{|Z_s|} \mathbf{1}_{\{Z_s < 0\}} dW_-^2(s) \right. \\
& + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u| > 1} \mathbf{1}_{\{Z_{s-} < 0\}} Z_{s-} (e^u - 1) \mathcal{N}_-^2(ds, dr, du) \\
& + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{|u| \leq 1} \mathbf{1}_{\{Z_{s-} < 0\}} Z_{s-} (e^u - 1) (\mathcal{N}_-^2 - \mathcal{N}_-^{2'}) (ds, dr, du) \\
& \left. + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-} < 0\}} Z_{s-} (-e^v - 1) \mathcal{M}_-^2(ds, dr, dv) \right], \quad t \leq T_0.
\end{aligned}$$

The final step is only for notational convenience: We change the coordinates for the jumps of \mathcal{N}_\pm^2 , \mathcal{M}_\pm^2 in order to combine the integrals to integrals driven by \mathcal{N}_\pm as in the statement of the Proposition. \square

2.2. Construction of Real-Valued Self-Similar Processes. The aim of this section is to construct real-valued self-similar Markov processes that leave zero continuously with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) whenever Condition (1.9) is valid. We construct a symmetric approximating sequence for the martingale problem corresponding to the SDE (1.7) and use moment equations of Bertoin and Yor [8] to show that limit points are Markovian and self-similar.

Recall that for a generator \mathcal{A} defined on a suitably chosen subset $\mathcal{D}(\mathcal{A})$ of the bounded and measurable functions $B(\mathbb{R})$ mapping \mathbb{R} into \mathbb{R} a stochastic process Z is said to be a solution to the martingale problem (\mathcal{A}, ν) corresponding to \mathcal{A} with initial distribution ν if for all $f \in \mathcal{D}(\mathcal{A})$

$$M_t^f = f(Z_t) - \int_0^t \mathcal{A}f(Z_s) ds, \quad t \geq 0,$$

is a martingale and Z_0 is distributed according to ν . The next proposition is standard; it is included for completeness and since the used estimates will appear several times in the sequel.

Proposition 2.4. *A stochastic process Z is a weak solution to the SDE (1.7) issued from $z \in \mathbb{R}$ if and only if it satisfies the martingale problem (\mathcal{A}, δ_z) corresponding to the generator*

$$\begin{aligned}
(\mathcal{A}f)(z) := & \left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) \text{sign}(z)f'(z) + \frac{\sigma^2}{2}|z|f''(z) \\
& + \int_0^\infty \int_{[-1,1]} \mathbf{1}_{\{|r|z| \leq 1\}} \left(f(uz) - f(z) - f'(z)z(u - 1) \right) dr \bar{\Pi}(du), \quad z \in \mathbb{R},
\end{aligned} \tag{2.8}$$

acting on the infinitely differentiable functions with compact support $C_c^\infty(\mathbb{R})$.

Proof. Let us first suppose Z is a weak solution to the SDE (1.7). Applying Itô's formula with $f \in C_c^\infty(\mathbb{R})$ yields

$$\begin{aligned} M_t^f &= f(Z_t) - f(z) - \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t f'(Z_s) \operatorname{sign}(Z_s) ds - \frac{\sigma^2}{2} \int_0^t f''(Z_s) |Z_s| ds \\ &\quad - \int_0^t \int_0^\infty \int_{-1}^1 (f(Z_s + \mathbf{1}_{\{|r|Z_s| \leq 1\}} Z_s(u-1)) - f(Z_s) - f'(Z_s) \mathbf{1}_{\{|r|Z_s| \leq 1\}} Z_s(u-1)) ds dr \bar{\Pi}(du) \\ &= f(Z_t) - f(z) - \int_0^t \mathcal{A}f(Z_s) ds \end{aligned}$$

is a local martingale, where

$$\begin{aligned} M_t^f &= \sigma \int_0^t f'(Z_s) \sqrt{|Z_s|} dB_s \\ &\quad + \int_0^t \int_0^\infty \int_{-1}^1 \mathbf{1}_{\{|r|Z_{s-}| \leq 1\}} (f(Z_{s-} + Z_{s-}(u-1)) - f(Z_{s-})) (\mathcal{N} - \mathcal{N}')(ds, dr, du). \end{aligned}$$

Moreover, it is easy to see that M^f is a true martingale. Indeed, by Theorem 51 of [30] it suffices to verify $E[\sup_{t \leq T} |M_t^f|] < \infty$ for all $T > 0$. Applying the Burkholder-Davis-Gundy inequality (for the non-continuous martingale see [16], p. 287) and the simple estimate $E[\sup_{t \leq T} |M_t^f|] \leq 1 + E[\sup_{t \leq T} |M_t^f|^2]$, we obtain

$$\begin{aligned} E[\sup_{t \leq T} |M_t^f|] &\leq 1 + 2\sigma^2 E \left[\sup_{t \leq T} \left| \int_0^t f'(Z_s) \sqrt{|Z_s|} dB_s \right|^2 \right] \\ &\quad + 2E \left[\sup_{t \leq T} \left| \int_0^t \int_0^\infty \int_{[-1,1]}^{\frac{1}{|Z_{s-}|}} (f(Z_{s-} + Z_{s-}(u-1)) - f(Z_{s-})) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \right|^2 \right] \\ &\leq 1 + CE \left[\int_0^T f'(Z_s)^2 |Z_s| ds \right] \\ &\quad + CE \left[\int_0^T \int_0^\infty \int_{[-1,1]}^{\frac{1}{|Z_{s-}|}} (f(Z_s + Z_s(u-1)) - f(Z_s))^2 ds dr \bar{\Pi}(du) \right]. \end{aligned}$$

By Taylor's formula and the boundedness of f' we find the upper bound

$$E[\sup_{t \leq T} |M_t^f|] \leq 1 + \left(\sup_z f'(z) \right)^2 \left(C + C \int_{[-1,1]} (u-1)^2 \bar{\Pi}(du) \right) E \left[\int_0^T |Z_s| ds \right]. \quad (2.9)$$

Note that the definition of $\bar{\Pi}$ implies

$$\begin{aligned} \int_{-1}^1 (u-1)^2 \bar{\Pi}(du) &= \int_{-1}^0 (u-1)^2 V(du) + \int_{-\infty}^0 (e^u - 1)^2 \Pi(du) \\ &\leq 2V([-1, 0)) + \Pi((-\infty, -1]) + C \int_{-1}^0 u^2 \Pi(du) \end{aligned}$$

which is finite since $V(du)$ is a finite measure and Π is a Lévy measure.

Next, we show that $E[\int_0^T |Z_s| ds]$ is finite. From Itô's isometry, Taylor's theorem and the estimate

$(a_1 + \dots + a_n)^2 \leq n(a_1^2 + \dots + a_n^2)$ for $a_i \in \mathbb{R}$ we get

$$\begin{aligned}
& E[|Z_t|] \\
& \leq 1 + E[|Z_t|^2] \\
& \leq 1 + 4z^2 + 4 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 t^2 + 4\sigma^2 E \left[\int_0^t |Z_s| ds \right] \\
& \quad + 4E \left[\int_0^t \int_0^{\frac{1}{|Z_s|}} \int_{-1}^1 Z_s^2 (u-1)^2 ds dr \bar{\Pi}(du) \right] \\
& \leq 1 + 4z^2 + 4 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 t^2 + 4 \left(\sigma^2 + \int_{-1}^1 (u-1)^2 \bar{\Pi}(du) \right) \int_0^T E[|Z_s|] ds.
\end{aligned}$$

Hence, Gronwall's inequality implies that $E[|Z_t|]$ grows at most exponentially so that $E[\int_0^T |Z_s| ds]$ is finite by Fubini's theorem. Now we can deduce from (2.9) that M_t^f is a martingale and the first part of the proof is complete.

Conversely, suppose the law of the process Z is a solution to the martingale problem (\mathcal{A}, δ_z) . By a standard stopping time argument to allow for the test-function $f(z) = z$, we have

$$Z_t = z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}(Z_s) ds + M_t, \quad t \geq 0,$$

almost surely, for a square-integrable martingale M that we have to identify. Let $\mathcal{C}(ds, dz)$ be the optional random measure on $[0, \infty) \times \mathbb{R}$ defined by the jumps of Z :

$$\mathcal{C}(ds, dz) = \sum_{s>0} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \delta_{(s, \Delta Z_s)}(ds, dz),$$

where $\Delta Z_s = Z_s - Z_{s-}$ is the jump of Z at time s . If \mathcal{C}' denotes the predictable compensator of \mathcal{C} , then page 376 of [16] shows that

$$Z_t = z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}(Z_s) ds + M_t^c + M_t^d \quad (2.10)$$

for a continuous martingale M^c and

$$M_t^d = \int_0^t \int_{\mathbb{R}} z (\mathcal{C} - \mathcal{C}')(ds, dz).$$

We now have to identify the martingales M^c and M^d . Applying Itô's formula to the semimartingale representation (2.10) of Z yields

$$\begin{aligned}
f(Z_t) &= f(z) + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t f'(Z_s) \text{sign}(Z_s) ds + \frac{1}{2} \int_0^t f''(Z_s) d[M_s^c, M_s^c] \\
&\quad + \int_0^t (f(Z_s + z) - f(Z_s) - f'(Z_s)z) \mathcal{C}'(ds, dz) + \text{local martingale}
\end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R})$. We can assume without loss of generality that the local martingale is a martingale since otherwise the rest of the proof can be carried out via localization. Comparing with the martingale problem (\mathcal{A}, δ_z) from (2.8) and using the uniqueness of the canonical decomposition for a semimartingale, we see that $d[M_s^c, M_s^c] = \sigma^2 |Z_s| ds$ and

$$\int_0^t \int_{\mathbb{R}} F(s, z) \mathcal{C}'(ds, dz) = \int_0^t \int_0^\infty \int_{-1}^1 F(s, \mathbf{1}_{\{r|Z_s| \leq 1\}} Z_s (u-1)) ds dr \bar{\Pi}(du).$$

for any non-negative Borel function F on $[0, \infty) \times \mathbb{R}$. Then we can find a Brownian motion B and an independent Poisson point process \mathcal{N} on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ by applying martingale representation theorems to (2.10) (see for instance [21] page 84 and page 93). \square

The construction of a solution to the martingale problem (\mathcal{A}, δ_z) is achieved with a series of lemmas. To give a rough idea how to construct solutions let us reconsider the simplest special case

$$dZ_t = \text{sign}(Z_t) dt + 2\sqrt{|Z_t|} dB_t, \quad Z_0 = 0, \quad (2.11)$$

and it's positive analogue

$$dZ_t = dt + 2\sqrt{Z_t} dB_t, \quad Z_0 = 0, \quad (2.12)$$

obtained for the absolute value. If W is a Brownian motion, then we already noted that $Z_t^{(1)} = \text{sign}(W_t)W_t^2$ is a weak solution to the SDE (2.11) and furthermore $Z_t^{(2)} = W_t^2$ is a weak solution to the SDE (2.12). Of course, $Z^{(1)}$ and $Z^{(2)}$ have a straight forward connection: given $Z^{(2)}$, $Z^{(1)}$ is obtained by reflecting every excursion at the origin with probability $1/2$.

An analogous procedure could be applied to construct symmetric solutions for the jump-type SDE (1.7) since excursion theory for $|Z|$ exists ($|Z|$ is a positive self-similar Markov process). In the general case of the SDE (1.7) solutions additionally jump over zero so that a direct modification of the reflection idea seems not to work. In what follows we give a stochastic calculus construction that mimics the reflection idea but is robust enough to encounter jumps that change signs.

The notations of Theorem 1.4 and Proposition 2.4 will be used in the sequel without explicit repetitions.

Lemma 2.5. *Suppose $m \in \mathbb{N}$ and that \mathcal{M}^m is a Poisson point process on $(0, \infty) \times \{-\frac{1}{m}, \frac{1}{m}\}$ independent of B and \mathcal{N} with intensity measure $\mathcal{M}^m(ds, dv) = ds \otimes \Sigma(dv)$, where $\Sigma(\{\frac{1}{m}\}) = \Sigma(\{-\frac{1}{m}\}) = \frac{m}{2}$. If we define*

$$\text{sign}_{(0)}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}},$$

then there are unique strong solutions Z^m to the SDE

$$\begin{aligned} Z_t = z &+ \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}_{(0)}(Z_s) ds + \sigma \int_0^t \sqrt{|Z_s|} dB_s \\ &+ \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right) \int_0^t \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}=0\}} v \mathcal{M}^m(ds, dv) \\ &+ \int_0^t \int_0^{\frac{1}{|Z_{s-}|} \wedge m} \int_{[-1, 1 - \frac{1}{m}]} Z_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0. \end{aligned} \quad (2.13)$$

Proof of Lemma 2.5. Suppressing the jumps according to the point process \mathcal{N} and integrating out dr in the remaining compensator integral yields the SDE

$$\begin{aligned} Z_t = z &+ \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}_{(0)}(Z_s) ds + \sigma \int_0^t \sqrt{|Z_s|} dB_s \\ &+ \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right) \int_0^t \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}=0\}} v \mathcal{M}^m(ds, dv) \\ &- \int_{[-1, 1 - \frac{1}{m}]} (u-1)\bar{\Pi}(du) \int_0^t \text{sign}(Z_s)(1 \wedge m|Z_s|) ds, \end{aligned}$$

where we used $\frac{x}{|x|} = \text{sign}(x)$ for $x \neq 0$. Whenever a solution is bounded away from zero, pathwise uniqueness and strong existence holds due to the local Lipschitz property of the integrands away from zero. When the solution hits zero it remains until it jumps according to a jump of \mathcal{M}^m . Since both jump integrals only jump with bounded rate, a strong solution of (2.13) can be constructed piecewise via the interlacing method.

This is a standard argument, so we omit the details and refer for instance to the proof of Proposition 3.5 of [17]. \square

The choice of the SDE (2.13) is motivated by the reflected excursion idea for a construction of solutions: whenever solutions are away from zero, they follow the original SDE (1.7), the "pseudo excursions" taking values in \mathbb{R}_* . At zero the pseudo excursions stop and after an exponential time a new pseudo excursion is started at a small initial state chosen symmetrically by \mathcal{M}^m . The symmetric restarting is needed to construct a symmetric process; non-symmetric restarting might be used to construct skew-self-similar Markov processes.

It is crucial to redefine the sign-function to be zero at zero since otherwise the constructed process is not a solution to (2.13). As m increases, the times between pseudo excursions and the new initial states tend to zero so that possible limiting processes leave zero continuously.

The construction shows that, for all z and m ,

$$\int_0^\infty \mathbf{1}_{\{Z_s^m=0\}} ds = \infty, \quad a.s.,$$

if the pseudo excursions hit zero in finite time. Hence, a priori it is possible that any limiting process Z is trapped at zero. To guarantee that Z is not such a trivial solution, under Condition (1.9) we are going to deduce

$$\int_0^\infty \mathbf{1}_{\{Z_s=0\}} ds = 0, \quad a.s. \quad (2.14)$$

In order to be able to verify (2.14), the constant in front of the stochastic integral with respect to \mathcal{M}^m turns out to be crucial. This might be surprising since in the limit $m \rightarrow \infty$ this integral vanishes without leaving a compensator term since it is a martingale. In order to show (2.14), we show in Lemma 2.8 below that the limiting absolute value $|Z|$ solves an SDE with constant drift. It is not clear a priori that the absolute value has constant drift since in (2.13) the drift is zero at zero. To ensure that the drift for the absolute value is constant (and not zero at zero) we use that the stochastic integral with respect to \mathcal{M}^m leaves in the absolute value the compensator integral

$$\begin{aligned} & \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m=0\}} ds \\ &= \left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m=0\}} ds + \left(\int_{-1}^0 (|u| - u)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m=0\}} ds. \end{aligned}$$

The summands compensate the time spend at zero that is not taken into account by $\text{sign}_{(0)}$ for the drift and $\text{sign}_{(0)}$ that appears as limit of $(\frac{1}{|Z_s|} \wedge m)Z_s$ for the compensated integral with respect to \mathcal{N} .

Lemma 2.6. *Suppose Z^m is as in Lemma 2.5, then*

$$\begin{aligned} |Z_t^m| &= |z| + \left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) t \\ &+ \left(\int_{-1}^0 (|u| - u)V(du) \right) \int_0^t (\mathbf{1}_{\{Z_s^m=0\}} + (1 \wedge m|Z_s^m|)) ds \\ &+ \sigma \int_0^t \text{sign}(Z_s^m) \sqrt{|Z_s^m|} dB_s \\ &+ \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) \int_0^t \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_s^m=0\}} |v| (\mathcal{M}^m - \mathcal{M}^{m'})(ds, dv) \\ &+ \int_0^t \int_0^{\frac{1}{|Z_s^m|} \wedge m} \int_{[-1, 1 - \frac{1}{m}]} |Z_s^m| (|u| - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0, \end{aligned} \quad (2.15)$$

almost surely.

Proof. Let us denote by $\tau_1 < \tau_2 < \dots$ the jumps of the Poissonian integral driven by \mathcal{M}^m which are precisely the times when Z^m leaves zero. Further, $\delta_1 < \delta_2 < \dots$ denote the successive first hitting

times of zero which do not accumulate since paths are càdlàg and solutions only leave zero with a jump of size $\pm \frac{\Psi(1) + \int_{-1}^0 (|u| - 1)V(du)}{m}$. If we define $ZERO := [\delta_1, \tau_1) \cup [\delta_2, \tau_2) \cup \dots$, then

$$Z_s^m = 0 \quad \forall s \in ZERO \quad \text{and} \quad Z_s^m \neq 0 \quad \forall s \notin ZERO.$$

Consequently, $|Z_s^m| = 0$ for $s \in ZERO$ so that it suffices to apply Tanaka's formula to Z^m on $ZERO^c$. Let us first show that the semimartingale local time at zero vanishes. The truncation by m implies that jumps are summable so that Corollary 3 on page 178 of [30] yields

$$\begin{aligned} L_t^0 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{|Z_s^m| \leq \varepsilon\}} d[Z_s^m, Z_s^m]^c \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\sigma^2}{\varepsilon} \left[\sum_{j=1}^{i-1} \int_{\tau_j}^{\delta_{j+1}} \mathbf{1}_{\{|Z_s^m| \leq \varepsilon\}} |Z_s^m| ds + \int_{\tau_i}^t \mathbf{1}_{\{|Z_s^m| \leq \varepsilon\}} |Z_s^m| ds \right] \\ &\leq \lim_{\varepsilon \rightarrow 0} \sigma^2 \left[\sum_{j=1}^{i-1} \int_{\tau_j}^{\delta_{j+1}} \mathbf{1}_{\{|Z_s^m| \leq \varepsilon\}} ds + \int_{\tau_i}^t \mathbf{1}_{\{|Z_s^m| \leq \varepsilon\}} ds \right], \quad t \in [\tau_i, \delta_{i+1}). \end{aligned}$$

Using dominated convergence, the righthand side converges to zero since Z^m does not spend time at zero on $ZERO^c$. Next, Tanaka's formula can be applied without additional local time term to deduce the semimartingale decomposition

$$\begin{aligned} &|Z_t^m| \\ &= |z| + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}(Z_s^m) \text{sign}_{(0)}(Z_s^m) ds + \sigma \int_0^t \text{sign}(Z_s^m) \sqrt{|Z_s^m|} dB_s \\ &\quad + \int_0^t \int_{\{\pm \frac{1}{m}\}} \left(\left| Z_{s-}^m + \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) \mathbf{1}_{\{Z_{s-}^m = 0\}} v \right| - |Z_{s-}^m| \right) \mathcal{M}^m(ds, dv) \\ &\quad + \int_0^t \int_0^{\frac{1}{|Z_{s-}^m|} \wedge m} \int_{[-1, 1 - \frac{1}{m}]} |Z_{s-}^m| (|u| - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du) \\ &\quad + \int_0^t (1 \wedge m|Z_s^m|) ds \int_{[-1, 1 - \frac{1}{m}]} (|u| - u) \bar{\Pi}(du). \end{aligned}$$

Adding and subtracting the compensator integral for \mathcal{M}^m , we obtain as a drift

$$\begin{aligned}
& \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m \neq 0\}} ds \\
& + \int_0^t \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m = 0\}} \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right) v \mathcal{M}^{m'}(ds, dv) \\
& + \int_0^t (1 \wedge m|Z_s^m|) ds \int_{[-1, 1 - \frac{1}{m}]} (|u|-u) \bar{\Pi}(du) \\
& = \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m \neq 0\}} ds \\
& + \left(\left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) + \left(\int_{-1}^0 (|u|-u)V(du) \right) \right) \int_0^t \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m = 0\}} |v| \Sigma(dv) ds \\
& + \int_0^t (1 \wedge m|Z_s^m|) ds \int_{[-1, 1 - \frac{1}{m}]} (|u|-u) \bar{\Pi}(du) \\
& = \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) t + \left(\int_{-1}^0 (|u|-u)V(du) \right) \int_0^t \mathbf{1}_{\{Z_s^m = 0\}} ds \\
& + \int_0^t (1 \wedge m|Z_s^m|) ds \int_{[-1, 1 - \frac{1}{m}]} (|u|-u) \bar{\Pi}(du).
\end{aligned}$$

Using the definition of $\bar{\Pi}$ we can simplify the final integral to

$$\int_{[-1, 1 - \frac{1}{m}]} (|u|-u) \bar{\Pi}(du) = \int_{-1}^0 (|u|-u) \bar{\Pi}(du) = \int_{-1}^0 (|u|-u)V(du)$$

from which the claim follows. \square

Next, we show that there are limits of the sequence Z^m :

Lemma 2.7. *For any $z \in \mathbb{R}$ the sequence $(Z^m)_{m \in \mathbb{N}}$ constructed in Lemma 2.5 is tight in the Skorokhod topology on \mathbb{D} .*

Proof. For the proof we apply Aldous's tightness criterion (see Aldous [1]). According to Aldous, to prove that $\{Z^m : m \in \mathbb{N}\}$ is tight in \mathbb{D} it is enough to show that

- (i) for every fixed $t \geq 0$, the set of random variables $\{Z_t^m : m \in \mathbb{N}\}$ is tight,
- (ii) for every sequence of stopping times $(\tau_m)_{m \in \mathbb{N}}$ (with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$) bounded above by $T > 0$ and for every sequence of positive real numbers $(\delta_m)_{m \in \mathbb{N}}$ converging to 0, $Z_{\tau_m + \delta_m}^m - Z_{\tau_m}^m \rightarrow 0$ in probability as $m \rightarrow \infty$.

To prove (i), by Markov's inequality it is enough to check that, for every fixed $t \geq 0$,

$$\sup_{m \in \mathbb{N}} E[(Z_t^m)^2] < \infty. \tag{2.16}$$

Using that $(a + b + c + d + e)^2 \leq 5(a^2 + b^2 + c^2 + d^2 + e^2)$ for $a, b, c, d, e \in \mathbb{R}$, we obtain that $E[(Z_t^m)^2]$ can be bounded by

$$\begin{aligned}
& 5z^2 + 5 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 E \left[\int_0^t \text{sign}_{(0)}(Z_s^m) ds \right]^2 + 5\sigma^2 E \left[\int_0^t \sqrt{|Z_s^m|} dB_s \right]^2 \\
& + 5 \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right)^2 E \left[\int_0^t \int_{\{-\frac{1}{m}, \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m = 0\}} v (\mathcal{M}^m - \mathcal{M}^{m'})(ds, dv) \right]^2 \\
& + 5E \left[\int_0^t \int_0^{\frac{1}{|Z_{s-}^m|} \wedge m} \int_{[-1, 1 - \frac{1}{m}]} Z_{s-}^m (u-1) (\mathcal{N} - \mathcal{N}')(ds, dr, du) \right]^2
\end{aligned}$$

which, via Itô's isometry (for the Poissonian integral see for instance page 62 in Ikeda and Watanabe [21]), can be bounded from above by

$$\begin{aligned}
& 5z^2 + 5 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 t^2 + 5\sigma^2 E \left[\int_0^t |Z_s^m| ds \right] \\
& + 5t \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right)^2 \int_{\{-\frac{1}{m}, \frac{1}{m}\}} v^2 \Sigma(dv) \\
& + 5E \left[\int_0^t \int_0^{\frac{1}{|Z_s^m|}} \int_{[-1, 1-\frac{1}{m}]} (Z_s^m)^2 (u-1)^2 ds dr \bar{\Pi}(du) \right] \\
& \leq 5z^2 + 5 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 t^2 + 5 \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right)^2 \frac{t}{m} \\
& + \left(5\sigma^2 + 5 \int_{-1}^1 (u-1)^2 \bar{\Pi}(du) \right) \int_0^t E[|Z_s^m|] ds
\end{aligned}$$

so that the estimate $E[|Z_s^m|] \leq 1 + E[(Z_s^m)^2]$ combined with Gronwall's inequality yields the claim.

Now we turn to (ii) proving the stronger statement that $Z_{\tau_m + \delta_m}^m - Z_{\tau_m}^m$ converges to 0 in L^2 as $m \rightarrow \infty$. Namely, by the SDE (2.13) and the splitting of summands as before,

$$\begin{aligned}
& E[|Z_{\tau_m + \delta_m}^m - Z_{\tau_m}^m|^2] \\
& \leq 4 \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right)^2 E \left[\int_{\tau_m}^{\tau_m + \delta_m} \text{sign}_{(0)}(Z_s^m) ds \right]^2 + 4\sigma^2 E \left[\int_{\tau_m}^{\tau_m + \delta_m} \sqrt{|Z_s^m|} dB_s \right]^2 \\
& + 4 \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right)^2 E \left[\int_{\tau_m}^{\tau_m + \delta_m} \int_{\{-\frac{1}{m}, \frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m = 0\}} v (\mathcal{M}^m - \mathcal{M}^{m'}) (ds, dv) \right]^2 \\
& + 4E \left[\int_{\tau_m}^{\tau_m + \delta_m} \int_0^{\frac{1}{|Z_{s-}^m|} \wedge m} \int_{[-1, 1-\frac{1}{m}]} Z_{s-}^m (u-1) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \right]^2.
\end{aligned}$$

The first summand can be estimated by $C\delta_m^2$ and, hence, can be neglected. By Proposition 3.2.10 in Karatzas and Shreve [23] we obtain

$$E \left[\int_{\tau_m}^{\tau_m + \delta_m} \sqrt{|Z_s^m|} dB_s \right]^2 = E \left[\int_{\tau_m}^{\tau_m + \delta_m} |Z_s^m| ds \right],$$

and, by Theorem II.1.33 in Jacod and Shiryaev [22], the optimal stopping theorem, using the same arguments as in the proof of (3.2.22) in Karatzas and Shreve [23], yields

$$\begin{aligned}
& E \left[\int_{\tau_m}^{\tau_m + \delta_m} \int_0^{\frac{1}{|Z_{s-}^m|} \wedge m} \int_{[-1, 1-\frac{1}{m}]} Z_{s-}^m (u-1) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \right]^2 \\
& \leq E \left[\int_{\tau_m}^{\tau_m + \delta_m} \int_0^{\frac{1}{|Z_s^m|}} \int_{[-1, 1-\frac{1}{m}]} (Z_s^m)^2 (u-1)^2 ds dr \bar{\Pi}(du) \right] \\
& \leq E \left[\int_{\tau_m}^{\tau_m + \delta_m} |Z_s^m| ds \right] \int_{-1}^1 (u-1)^2 \bar{\Pi}(du).
\end{aligned}$$

For the integral with respect to \mathcal{M} a similar arguments gives the upper bound $C_3\delta_m$. In total this shows that (we can suppose that $\delta_m \leq 1$),

$$\begin{aligned}
E[|Z_{\tau_m + \delta_m}^m - Z_{\tau_m}^m|^2] & \leq C_1\delta_m^2 + C_2\delta_m + C_3E \left[\int_{\tau_m}^{\tau_m + \delta_m} |Z_s^m| ds \right] \\
& \leq C_1\delta_m^2 + C_2\delta_m + C_3\delta_m E \left[\sup_{t \leq T+1} |Z_t^m| \right].
\end{aligned}$$

Hence, the proof is complete if we can show that $E[\sup_{t \leq T+1} |Z_s^m|]$ is bounded in m . But this follows easily with the same arguments exploited for (i) using the Burkholder-Davis-Gundy inequality instead of Itô's isometry. \square

We next prepare for the convergence proof of Z^m along subsequences. It is crucial to deduce, a priori, that all limiting points do not spend time at zero in order to control the discontinuity of the sign-function at zero. Since we cannot deduce this property for the limiting points of Z^m directly, we show it for $|Z^m|$ which is substantially simpler.

Lemma 2.8. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ constructed in Lemma 2.5, then $|Z|$ is a weak solution to the SDE*

$$\begin{aligned} X_t = z + & \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) t + \sigma \int_0^t \sqrt{X_s} dB_s \\ & + \int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 X_{s-} (|u| - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du). \end{aligned} \quad (2.17)$$

Proof. Let us suppose that along the subsequence m_k we have weak convergence of Z^{m_k} to Z and, due to the continuous mapping theorem, also weak convergence of $|Z^{m_k}|$ to $|Z|$. We first derive a martingale problem for $|Z^m|$, $m \in \mathbb{N}$, from which we then derive the claimed statement.

Step A): Proceeding exactly as in Step 1) of the proof of Proposition 2.4, one derives from Lemma 2.6 that $|Z^m|$ solves the martingale problem $(|\mathcal{A}^m|, \delta_{|z|})$ with

$$\begin{aligned} & (|\mathcal{A}^m|f)(x) \\ := & \left(\left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) \right. \\ & + \left. \left(\int_{-1}^0 (|u| - u)V(du) \right) (\mathbf{1}_{\{x=0\}} + (1 \wedge mx)) \right) f'(x) \\ & + \frac{\sigma^2}{2} x f''(x) + \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{x=0\}} \left(f \left(\left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) |u| \right) \right. \\ & \quad \left. - f(0) - f'(0) \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) |u| \right) \Sigma(dv) \\ & + \int_0^m \int_{[-1, 1 - \frac{1}{m}]} \mathbf{1}_{\{rx \leq 1\}} \left(f(|u|x) - f(x) - f'(x)x(|u| - 1) \right) dr \bar{\Pi}(du), \quad x \geq 0, \end{aligned}$$

for $f \in C_c^\infty[0, \infty)$.

Step B): Next, we show that the limit points $|Z|$ solve the martingale problem $(|\mathcal{A}|, \delta_{|z|})$, with

$$\begin{aligned} (|\mathcal{A}|f)(x) := & \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) f'(x) + \frac{\sigma^2}{2} x f''(x) \\ & + \int_0^\infty \int_{[-1, 1]} \mathbf{1}_{\{rx \leq 1\}} \left(f(|u|x) - f(x) - f'(x)x(|u| - 1) \right) dr \bar{\Pi}(du), \quad x \geq 0, \end{aligned}$$

for $f \in C_c^\infty[0, \infty)$. By Skorokhod's representation theorem we may assume that Z^{m_k} converges to Z (resp. $|Z^{m_k}|$ to $|Z|$) almost surely in \mathbb{D} (possibly on a different probability space and changing also the subsequence $(m_k)_{k \in \mathbb{N}}$). By Proposition 3.5.2 of [18], the almost sure convergence yields that $P(\bar{\Omega}) = 1$, where

$$\bar{\Omega} := \left\{ \omega \in \Omega : \lim_{k \rightarrow \infty} |Z_t^{m_k}(\omega)| = |Z_t(\omega)| \quad \text{for } t \geq 0 \text{ at which } (Z_u(\omega))_{u \geq 0} \text{ is continuous} \right\}.$$

In what follows we let $\omega \in \bar{\Omega}$ be fixed and show that

$$\lim_{k \rightarrow \infty} \int_0^t (|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}|)(\omega) ds = \int_0^t (|\mathcal{A}|f)(|Z_s|)(\omega) ds, \quad t \geq 0. \quad (2.18)$$

In Step B1) we verify the pointwise convergence $(|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}|)(\omega) \xrightarrow{m_k \rightarrow \infty} (|\mathcal{A}|f)(|Z_s|)(\omega)$ for $s \leq t$ fixed and in Step B2) we verify the convergence of (2.18) via dominated convergence.

Let us introduce the notation

$$D(\omega) := \{t \geq 0 : (Z_u(\omega))_{u \geq 0} \text{ is continuous at } t\},$$

so that $\lim_{k \rightarrow \infty} |Z_t^{m_k}(\omega)| = |Z_t(\omega)|$ for all $t \in D(\omega)$ and furthermore $[0, \infty) \setminus D(\omega)$ is at most countable since Z has càdlàg paths.

Step B1a): The pointwise convergence for the drift part is trivial.

Step B1b): The pointwise convergence for the diffusive part is trivial.

Step B1c): For the integral with respect to Σ we apply Taylor's formula to find

$$\begin{aligned} & \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{Z_s^{m_k}=0\}} \left(f \left(\left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) |v| \right) \right. \\ & \quad \left. - f(0) - f'(0) \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) |v| \right) \Sigma(dv) \\ & \leq \frac{1}{2} \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right)^2 \sup_z f''(z) \int_{\{-\frac{1}{m_k}, \frac{1}{m_k}\}} |v|^2 \Sigma(dv) \\ & = \frac{1}{2} \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right)^2 \frac{\sup_z f''(z)}{m_k}, \end{aligned}$$

so that pointwise convergence to zero for $m_k \rightarrow \infty$ is verified.

Step B1d): For the integral with respect to $\bar{\Pi}$ we use that, for all $s \in D(\omega)$, the integrand

$$\begin{aligned} (r, u) & \mapsto \mathbf{1}_{\{u \in [-1, 1 - \frac{1}{m}]\}} \mathbf{1}_{\{r|Z_s^{m_k}(\omega)| \leq 1\}} \\ & \quad \times [f(|u||Z_s^{m_k}(\omega)|) - f(|Z_s^{m_k}(\omega)|) - f'(|Z_s^{m_k}(\omega)|)|Z_s^{m_k}(\omega)|(|u| - 1)] \end{aligned}$$

converges, as $k \rightarrow \infty$, pointwise to

$$(r, u) \mapsto \mathbf{1}_{\{|u| \leq 1\}} \mathbf{1}_{\{r|Z_s(\omega)| \leq 1\}} [f(|u||Z_s(\omega)|) - f(|Z_s(\omega)|) - f'(|Z_s(\omega)|)|Z_s(\omega)|(|u| - 1)].$$

Step B2): Since

$$\lim_{k \rightarrow \infty} (|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}(\omega)|) = (|\mathcal{A}|f)(|Z_s(\omega)|), \quad \omega \in \bar{\Omega},$$

is verified for any $s \leq t$, it remains to justify the change of limit and integration in (2.18). For the first three summands dominated convergence is clear (in Step B1c) the upper bound is independent of s) and we only need to deal with the integral with respect to $\bar{\Pi}$:

By Taylor expansion of second order, we can derive the upper bound

$$\begin{aligned} & \sup_{k \in \mathbb{N}} \mathbf{1}_{u \in [-1, 1 - \frac{1}{m}]} \mathbf{1}_{\{r|Z_s^{m_k}(\omega)| \leq 1\}} \\ & \quad \left| f(|u||Z_s^{m_k}(\omega)|) - f(|Z_s^{m_k}(\omega)|) \mathbf{1}_{\{|u| \geq \varepsilon_k\}} - f'(|Z_s^{m_k}(\omega)|)|Z_s^{m_k}(\omega)|(|u| - 1) \right| \\ & \leq \frac{1}{2} \mathbf{1}_{\{|u| \leq 1\}} \sup_z |f''(z)| \sup_{k \in \mathbb{N}} \mathbf{1}_{\{r|Z_s^{m_k}(\omega)| \leq 1\}} |Z_s^{m_k}(\omega)|^2 (|u| - 1)^2 \\ & \leq \frac{1}{2} \mathbf{1}_{\{|u| \leq 1\}} \sup_z |f''(z)| \sup_{k \in \mathbb{N}} \left(Z_s^{m_k}(\omega) \wedge \frac{1}{r} \right)^2 (|u| - 1)^2 \\ & \leq \frac{1}{2} \mathbf{1}_{\{|u| \leq 1\}} \sup_z |f''(z)| \sup_{k \in \mathbb{N}} \left(\sup_{s \leq t} Z_s^{m_k}(\omega) \wedge \frac{1}{r} \right)^2 (|u| - 1)^2. \end{aligned}$$

Note that for the final line we used that $x \mapsto \sup_{s \leq t} x_s$ is a continuous functional on the Skorokhod space so that the convergence of Z^{m_k} implies $\sup_{k \in \mathbb{N}} \sup_{s \leq t} Z_s^{m_k}(\omega) =: C_t(\omega) < \infty$. Thus, the integral with respect to $\bar{\Pi}$ in $(\mathcal{A}^{m_k} f)(|Z_s^{m_k}(\omega)|)$ is bounded from above by

$$\begin{aligned} & \frac{1}{2} \sup_z |f''(z)| \int_0^\infty \left(C_t(\omega) \wedge \frac{1}{r} \right)^2 dr \int_{-1}^1 (|u| - 1)^2 \bar{\Pi}(du) \\ & \leq C \sup_z |f''(z)| \left(C_t(\omega)^2 + \int_1^\infty r^{-2} dr \right) \int_{-1}^1 (|u| - 1)^2 \bar{\Pi}(du), \end{aligned}$$

which is finite and independent of s . Using dominated convergence theorem we have convergence for the third summand of $\mathcal{A}^{m_k} f$.

Step C): To conclude the proof let us write

$$\begin{aligned} M_t^{m_k} &= f(Z_t^{m_k}) - \int_0^t |\mathcal{A}^{m_k}| f(Z_s^{m_k}) ds, \quad t \geq 0, \\ M_t &= f(Z_t) - \int_0^t |\mathcal{A}| f(Z_s) ds, \quad t \geq 0, \end{aligned}$$

for which we know that the M^{m_k} are martingales with respect to the filtrations generated by Z^{m_k} . The martingale property of M with respect to its own filtration follows by Jacod and Shiryaev [22, Corollary IX.1.19]. To check the conditions of this result we have to show that M^{m_k} converges weakly in \mathbb{D} as $k \rightarrow \infty$ to M and that there is some $b \geq 0$ such that $|\Delta M_t^{m_k}| \leq b$ for all $t > 0$, $m \in \mathbb{N}$, almost surely. Using that Z^{m_k} converges weakly in \mathbb{D} to Z as $k \rightarrow \infty$ and that f is continuous and bounded, we have $f(Z^{m_k})$ converges weakly in \mathbb{D} to $f(Z)$ as $k \rightarrow \infty$. Since the integral in the definition of M is continuous, by Jacod and Shiryaev [22, Proposition VI.1.23], we obtain that M^{m_k} converges weakly in \mathbb{D} as $k \rightarrow \infty$ to M . Further, almost surely for all $t \geq 0$,

$$|\Delta M_t^{m_k}| = |f(|Z_t^{m_k}|) - f(|Z_{t-}^{m_k}|)| \leq 2 \sup_z |f(z)| < \infty.$$

□

Corollary 2.9. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ constructed in Lemma 2.5, then Z almost surely does not spend time at zero.*

Proof. Utilizing Lemma 2.8 it is enough to show that any non-negative weak solution to the SDE (2.17) does not spend time at zero. Without further assumptions on the jump measure $\bar{\Pi}$ the jumps of a solution X to the SDE (2.17) are not summable, thus, we cannot directly resort to a simple local time argument based on the occupation time formula (compare for instance Section IV.6 of [30]). Instead, we use an Itô formula argument that was used in a more specific situation in [4]. The argument is based on the trivial fact $\sqrt{X_t^2} = X_t$ and a double use of Itô's formula, once applied to a smooth function and once to a singular function. The singular use gives an additional term from which the claim follows. Here is the simple direction applying Itô's formula to the $C^2([0, \infty))$ -function $f(x) = x^2$:

$$\begin{aligned} X_t^2 &= z^2 + \left(2 \left(\Psi(1) + \int_{-1}^0 (|u| - 1) V(du) \right) + \sigma^2 + \int_{-1}^1 (u^2 - 2|u| + 1) \bar{\Pi}(du) \right) \int_0^t X_s ds \\ &\quad + 2\sigma \int_0^t X_s^{3/2} dB_s + \int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 X_{s-}^2 (u^2 - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0. \end{aligned}$$

Next, we proceed with the inverse direction. Suppose we could apply Itô's formula with $f(x) = \sqrt{x}$ and the convention $\frac{0}{0} = 0$ to the semimartingale decomposition derived for X_t^2 , then

$$\begin{aligned} X_t = z + & \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) \int_0^t \mathbf{1}_{\{X_s > 0\}} ds + \sigma \int_0^t \sqrt{X_s} dB_s \\ & + \int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 X_{s-}(|u| - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0, \end{aligned} \quad (2.19)$$

so that comparing the drifts of (2.19) and (2.17) implies the claim. To verify Equation (2.19) rigorously, we approximate $f(x) = \sqrt{x}$ on $[0, \infty)$ by the $C^2([0, \infty))$ -functions $f^\varepsilon(x) = \sqrt{x + \varepsilon}$. Applying Itô's formula to the semimartingale decomposition derived for X_t^2 gives

$$\begin{aligned} & \sqrt{X_t^2 + \varepsilon} \\ = & \sqrt{z^2 + \varepsilon} + \left(2 \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) + \sigma^2 + \int_{-1}^1 (u^2 - 2|u| + 1) \bar{\Pi}(du) \right) \frac{1}{2} \int_0^t (X_s^2 + \varepsilon)^{-\frac{1}{2}} X_s ds \\ & + \sigma \int_0^t (X_s^2 + \varepsilon)^{-\frac{1}{2}} X_s^{3/2} dB_s - \frac{\sigma^2}{2} \int_0^t (X_s^2 + \varepsilon)^{-\frac{3}{2}} X_s^3 ds \\ & + \int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 \left(\sqrt{X_{s-}^2 u^2 + \varepsilon} - \sqrt{X_{s-}^2 + \varepsilon} \right) (\mathcal{N} - \mathcal{N}')(ds, dr, du) \\ & + \int_0^t \int_0^{\frac{1}{X_s}} \int_{-1}^1 \left(\sqrt{X_s^2 u^2 + \varepsilon} - \sqrt{X_s^2 + \varepsilon} - \frac{1}{2} (X_s^2 + \varepsilon)^{-1/2} X_s^2 (u^2 - 1) \right) ds dr \bar{\Pi}(du) \\ = & \sqrt{z^2 + \varepsilon} + I_t^{1,\varepsilon} + I_t^{2,\varepsilon} + I_t^{3,\varepsilon} + I_t^{4,\varepsilon} + I_t^{5,\varepsilon}. \end{aligned}$$

Since the left-hand side converges to X_t almost surely, it suffices to find a subsequence ε_k along which the summands $I_t^{1,\varepsilon_k}, \dots, I_t^{5,\varepsilon_k}$ converge almost surely to the summands of (2.19).

For the drift we directly obtain the almost sure convergence

$$I_t^{1,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \left(\left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) + \frac{\sigma^2}{2} + \frac{1}{2} \int_{-1}^1 (u^2 - 2|u| + 1) \bar{\Pi}(du) \right) \int_0^t \mathbf{1}_{\{X_s > 0\}} ds$$

by dominated convergence. To show convergence of $I_t^{2,\varepsilon}$ we first use Itô's isometry to obtain

$$E \left[\left(I_t^{2,\varepsilon} - \sigma \int_0^t \sqrt{X_s} dB_s \right)^2 \right] = \sigma^2 E \left[\int_0^t ((X_s^2 + \varepsilon)^{-\frac{1}{2}} X_s^{3/2} - X_s^{1/2})^2 ds \right]. \quad (2.20)$$

If we define $g_\varepsilon(x) = ((x^2 + \varepsilon)^{-\frac{1}{2}} x^{3/2} - x^{1/2})^2$, then

$$\frac{\partial}{\partial \varepsilon} g_\varepsilon(x) = -((x^2 + \varepsilon)^{-\frac{1}{2}} x^{3/2} - x^{1/2})(x^2 + \varepsilon)^{-3/2} x^{3/2} \geq 0, \quad x \geq 0.$$

Since $g_\varepsilon(x)$ converges pointwise to zero as ε tends to zero, the righthand side of (2.20) converges to zero by monotone convergence so that $I_t^{2,\varepsilon}$ converges to $\sigma \int_0^t \sqrt{X_s} dB_s$ in L^2 . The almost sure convergence

$$I_t^{3,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} -\frac{\sigma^2}{2} \int_0^t \mathbf{1}_{\{X_s > 0\}} ds$$

is proved as in Step 1) and cancels in the limit the second summand of $I_t^{1,\varepsilon}$. To show convergence of $I_t^{4,\varepsilon}$ we use the Itô isometry for Poissonian integrals to find

$$\begin{aligned} & E \left[\left(I_t^{4,\varepsilon} - \int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 X_{s-}(|u| - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du) \right)^2 \right] \\ = & E \left[\int_0^t \int_{-1}^1 \frac{1}{X_s} \left(\sqrt{X_s^2 u^2 + \varepsilon} - \sqrt{X_s^2 + \varepsilon} - X_s(|u| - 1) \right)^2 ds \bar{\Pi}(du) \right]. \end{aligned}$$

With $h_\varepsilon(x) = \frac{1}{x}(\sqrt{x^2 u^2 + \varepsilon} - \sqrt{x^2 + \varepsilon} - x(|u| - 1))^2$ we claim that

$$\frac{\partial}{\partial \varepsilon} h_\varepsilon(x) = -\frac{1}{x}(\sqrt{x^2 u^2 + \varepsilon} - \sqrt{x^2 + \varepsilon} - x(|u| - 1))((x^2 u^2 + \varepsilon)^{-1/2} - (x^2 + \varepsilon)^{-1/2}) \leq 0$$

for all $x \geq 0$ and $|u| \leq 1$. To see the claim note that the second bracket is clearly positive so that it suffices to show that the first bracket is positive. For $\varepsilon = 0$ the first bracket is zero and it is easy to see that the first derivative in ε is positive. Hence, the L^2 -convergence of $I_t^{4,\varepsilon}$ to $\int_0^t \int_0^{\frac{1}{X_{s-}}} \int_{-1}^1 X_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du)$ follows again from monotone convergence.

Finally, if we rewrite $I_t^{5,\varepsilon}$ as

$$\begin{aligned} I_t^{5,\varepsilon} &= \int_0^t \int_{-1}^1 \frac{1}{X_s} (\sqrt{X_s^2 u^2 + \varepsilon} - \sqrt{X_s^2 + \varepsilon}) ds \bar{\Pi}(du) \\ &\quad - \int_0^t \int_{-1}^1 \frac{1}{2} (X_s^2 + \varepsilon)^{-1/2} X_s (u^2 - 1) ds \bar{\Pi}(du), \end{aligned}$$

then the first summand converges by monotone convergence shown as above and the second summand by dominated convergence:

$$I_t^{5,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \int_{-1}^1 \left(|u| - 1 - \frac{1}{2}(u^2 - 1) \right) \bar{\Pi}(du) \int_0^t \mathbf{1}_{\{X_s > 0\}} ds, \quad a.s.,$$

so that the third summand of $I^{1,\varepsilon}$ is cancelled in the limit.

Choosing a common subsequence ε_k such that all terms converge almost surely the proof can be completed. \square

With the previous lemma we know that under condition (1.9) solutions do not become trivial (trapped at zero). Additionally we can circumvent two major problems of the construction: First, the problem of the discontinuity of the sign-function at zero is resolved. Secondly, our redefinition of $\text{sign}_{(0)}$ in the approximating equations to ensure symmetry of the approximating sequence does not pose any problem since it is not seen by the limiting process Z .

Lemma 2.10. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ constructed in Lemma 2.5, then Z is a weak solution to (1.7).*

Proof. Taking into account Proposition 2.4 it suffices to show that any weak limiting point Z of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ constructed in Lemma 2.5 satisfies the martingale problem (\mathcal{A}, δ_z) . The proof follows along the same lines as the proof of Lemma 2.8 changing the state-space from $[0, \infty)$ to \mathbb{R} and the generators to

$$\begin{aligned} &(\mathcal{A}^m f)(x) \\ &:= \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \text{sign}_{(0)}(x) f'(x) + \frac{\sigma^2}{2} |x| f''(x) \\ &\quad + \int_{\{\pm \frac{1}{m}\}} \mathbf{1}_{\{x=0\}} \left[f \left(\left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right) v \right) \right. \\ &\quad \quad \left. - f(0) - f'(0) \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du) \right) v \right] \Sigma(dv) \\ &\quad + \int_0^m \int_{[-1, 1 - \frac{1}{m}]} \mathbf{1}_{\{|r|x| \leq 1\}} \left(f(ux) - f(x) - f'(x)x(u-1) \right) dr \bar{\Pi}(du), \quad f \in C_c^\infty((-\infty, \infty)), \end{aligned}$$

and

$$\begin{aligned}
& (\mathcal{A}f)(x) \\
& := \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \text{sign}(x)f'(x) + \frac{\sigma^2}{2}|x|f''(x) \\
& + \int_0^\infty \int_{[-1,1]} \mathbf{1}_{\{|r|x| \leq 1\}} \left(f(ux) - f(x) - f'(x)x(u-1) \right) dr \bar{\Pi}(du), \quad f \in C_c^\infty((-\infty, \infty)).
\end{aligned}$$

Comparing with the proof of Lemma 2.8, the only difference occurs in Step B1a) because the pointwise convergence for the drift coefficients fails since (i) the sign-function is defined differently for the approximating martingale problem and the limit martingale problem and (ii) both sign-functions are discontinuous. Both problems are avoidable via Corollary 2.9 applied for the final step of

$$\begin{aligned}
\lim_{m \rightarrow \infty} \int_0^t \text{sign}_{(0)}(Z_s^m(\omega)) ds &= \lim_{m \rightarrow \infty} \int_0^t \text{sign}_{(0)}(Z_s^m(\omega)) \mathbf{1}_{\{Z_s(\omega) \neq 0\}} ds \\
&= \int_0^t \text{sign}_{(0)}(Z_s(\omega)) \mathbf{1}_{\{Z_s(\omega) \neq 0\}} ds \\
&= \int_0^t \text{sign}(Z_s(\omega)) \mathbf{1}_{\{Z_s(\omega) \neq 0\}} ds \\
&= \int_0^t \text{sign}(Z_s(\omega)) ds.
\end{aligned}$$

For the pointwise convergence of the integrands we used the continuity of the sign-function away from zero and dominated convergence to interchange limits and integration. The proof is now complete. \square

Now that we have constructed processes Z^z started from $z \in \mathbb{R}$ that are weak solutions to (1.7) and symmetric by construction, we need to show that the family $(Z^z)_{z \in \mathbb{R}}$ is

- (i) Markovian,
- (ii) self-similar.

Both statements are derived from a weak uniqueness statement for (1.7) for which we had to impose Assumption **(A)**.

For initial condition zero, we derive moment equations for (1.7) from which, due to Assumption **(A)**, the well-posedness of the moment problem for one-dimensional marginals of symmetric solutions can be deduced. For initial conditions different from zero pathwise uniqueness before hitting zero holds. Combining the two uniqueness statements, uniqueness for one-dimensional marginals of solutions issued from the same initial condition follows. The Markov property is then a consequence of martingale problem theory and the self-similarity can be deduced from the self-similar structure of the coefficients in (1.7).

Proposition 2.11. *Denote by Z a limiting point of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ with initial conditions $Z_0^m = z$ constructed in Lemma 2.5, then Z is Markovian.*

Proof. Since we showed that Z is a weak solution to the SDE (1.7) we start with a weak uniqueness statement for solutions to the SDE (1.7) that satisfies the symmetry property

$$P(X_{T_0+t} \in A) = P(X_{T_0+t} \in -A), \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}), \quad (2.21)$$

and do not spend time at zero. Both properties hold for Z : the first by construction as weak limit of the symmetric Z^m defined by (2.13), the latter by Corollary 2.9.

Step 1a): Let us first deduce the almost sure dichotomy $T_0 < \infty$ or $T_0 = \infty$ for the first hitting time of zero. A singular application of Itô's formula as in the proof of Corollary 2.9 yields the

semimartingale decomposition

$$\begin{aligned} |X_t| &= |z| + \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) t \\ &\quad + \sigma \int_0^t \text{sign}(X_s) \sqrt{|X_s|} dB_s + \int_0^t \int_0^{\frac{1}{|X_{s-}|}} \int_{-1}^1 |X_{s-}| (|u| - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned} \quad (2.22)$$

for $t \geq 0$. Replacing B by the Brownian motion $\bar{B}_t = \int_0^t \text{sign}(X_s) dB_s$, we find that (2.22) coincides with (2.2) so that $|X|$ is a positive self-similar Markov process. Since the first hitting times of X and $|X|$ coincide, the claimed dichotomy follows from Lamperti's dichotomy (Section 3 of [27]) for positive self-similar Markov processes.

Step 1b): Pathwise uniqueness holds for (1.7) up to first hitting zero: Suppose that X^1, X^2 are two solutions driven by the same noises B and \mathcal{N} and set $T_{\frac{1}{n}} = \inf\{t \geq 0 : |X_t^1| \leq \frac{1}{n} \text{ or } |X_t^2| \leq \frac{1}{n}\}$ for $\frac{1}{n} < |X_0^i|$. Then,

$$P(X_t^1 = X_t^2 \text{ for all } t < T_{\frac{1}{n}}) = 1$$

since all integrands are locally Lipschitz continuous away from zero. Letting n tend to infinity and using the right-continuity of solutions we find that

$$P(X_t^1 = X_t^2 \text{ for all } t < T_0) = 1$$

and in particular that

$$T_0 = \inf\{t \geq 0 : X_t^1 = 0\} = \inf\{t \geq 0 : X_t^2 = 0\}.$$

In what follows we assume $T_0 < \infty$ almost surely; otherwise, we can directly proceed with Step 2) since pathwise uniqueness implies uniqueness in law.

Step 1c): The pathwise uniqueness implies that solutions to (1.7) are strong up to T_0 and, consequently, T_0 is a stopping time for B and \mathcal{N} . We denote by \tilde{B} and $\tilde{\mathcal{N}}$ the noises shifted by T_0 . Due to the strong Markov property of the Brownian motion and the Poisson point process \tilde{B} is a Brownian motion and $\tilde{\mathcal{N}}$ a Poisson point process with same intensity as \mathcal{N} . Furthermore, we define the shifted process $(\tilde{X}_t = X_{T_0+t})_{t \geq 0}$ that satisfies weakly the SDE

$$\begin{aligned} \tilde{X}_t &= \left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) \int_0^t \text{sign}(\tilde{X}_s) ds + \sigma \int_0^t \sqrt{|\tilde{X}_s|} d\tilde{B}_s \\ &\quad + \int_0^t \int_0^{\frac{1}{|\tilde{X}_{s-}|}} \int_{-1}^1 \tilde{X}_{s-} (u - 1) (\tilde{\mathcal{N}} - \tilde{\mathcal{N}}')(ds, dr, du), \quad t \geq 0. \end{aligned} \quad (2.23)$$

In other words, \tilde{X} is a weak solution to the SDE (1.7) with respect to the noises \tilde{B} and $\tilde{\mathcal{N}}$ issued from zero. Furthermore, the symmetry condition (2.21) implies the symmetry condition

$$P(\tilde{X}_t \in A) = P(\tilde{X}_t \in -A), \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}), \quad (2.24)$$

and clearly \tilde{X} does not spend time at zero since X does not.

Step 1d): Next, we need that all solutions to the SDE (2.23) that do not spend time at zero and satisfy the symmetry condition (2.24) have the same one-dimensional marginals. The symmetry assumption implies that the one-dimensional laws \tilde{X}_t are uniquely determined by $|\tilde{X}_t|$. Since $|\tilde{X}_t|$ satisfies the SDE (2.22) with the noises \tilde{B} and $\tilde{\mathcal{N}}$ it suffices to show that the moment problem for (2.22) is well-posed; this follows from the main result of [3] since the jumps of

$$t \mapsto \int_0^t \int_0^{\frac{1}{|\tilde{X}_{s-}|}} \int_{-1}^1 |\tilde{X}_{s-}| (|u| - 1) (\tilde{\mathcal{N}} - \tilde{\mathcal{N}}')(ds, dr, du)$$

are negative (this is the reason for our Assumption **(A)**). It was shown in [3] that the k th moments equal $C_k t^k$ for some constants C_k that decrease sufficiently fast so that the moment problem is well-posed. Furthermore, the C_k only depend on (a, σ^2, Π, q, V) but not on the solution, thus, well-posedness of the moment problem implies the uniqueness of one-dimensional marginals.

Step 1e): Let us now suppose X^1 and X^2 are two weak solutions to (1.7) that do not spend time at zero and both satisfy the symmetry condition (2.21). We split according to

$$P(X_t^i \in A) = P(X_t^i \in A, t \leq T_0^i) + P(X_t^i \in A, t > T_0^i)$$

and show that

$$P(X_t^1 \in A, t \leq T_0^1) = P(X_t^2 \in A, t \leq T_0^2), \quad (2.25)$$

$$P(X_t^1 \in A, t > T_0^1) = P(X_t^2 \in A, t > T_0^2). \quad (2.26)$$

Equality (2.25) follows from the pathwise uniqueness before hitting zero so that we only need to verify (2.26). Using the defining equation for the X^i and the definition of \tilde{X}^i above, one can rewrite

$$P(X_t^i \in A, t > T_0^i) = P(\tilde{X}_{t-T_0^i}^i \in A, t > T_0^i).$$

Integrating out $P(T_0^i \in ds)$ (note that $P(T_0^1 \in ds) = P(T_0^2 \in ds)$ have same law as shown in Step 1b)) we obtain (2.26) from Step 1d) since $\tilde{X}_0^i = 0$.

Step 2): The uniqueness of one-dimensional marginals for weak solutions to (1.7) that spend zero time at zero and satisfy the symmetry condition (2.24) now implies the Markov property for the weak solution Z by martingale problem arguments such as in the proof of Theorem 4.4.2 of [18]. The required measurability $z \mapsto P^z$ is a consequence of the construction: the measurability (even continuity) in the initial condition holds for the Z^m and since the pointwise limit of measurable functions remains measurable, the measurability for the limit follows. \square

Proposition 2.12. *Denote by Z^z a limiting point of the tight sequence $(Z^m)_{m \in \mathbb{N}}$ with initial conditions $z \in \mathbb{R}$ constructed in Lemma 2.5. Then the family $(Z^z)_{z \in \mathbb{R}}$ is a real-valued self-similar Markov family with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) .*

Proof. For $c > 0$ fixed we define $\bar{Z}_t^z := \frac{1}{c} Z_{ct}^z, t \geq 0$. Since Z is a weak solution to (1.7), \bar{Z}^z satisfies

$$\begin{aligned} \bar{Z}_t^z &= \frac{z}{c} + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \frac{1}{c} \int_0^{ct} \text{sign}(Z_s^z) ds + \frac{\sigma}{c} \int_0^{ct} \sqrt{|Z_s^z|} dB_s \\ &\quad + \frac{1}{c} \int_0^{ct} \int_0^{\frac{1}{|Z_{s-}^z|}} \int_{-1}^1 Z_{s-}^z (u-1) (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0, \end{aligned}$$

almost surely. By Revuz and Yor [31, Proposition V.1.5] we have the almost sure identity

$$\frac{\sigma}{c} \int_0^{ct} \sqrt{|Z_s^z|} dB_s = \sigma \int_0^t \sqrt{c^{-1}|Z_{cs}^z|} d(c^{-1/2} B_{cs}), \quad t \geq 0.$$

Next, we use the analogue almost sure identity

$$\begin{aligned} &\int_0^{ct} \int_0^{\frac{1}{|Z_{s-}^z|}} \int_{-1}^1 Z_{s-}^z (u-1) (\mathcal{N} - \mathcal{N}')(ds, dr, du) \\ &= \int_0^t \int_0^{\frac{1}{c|Z_{cs-}^z|}} \int_{-1}^1 Z_{cs-}^z (u-1) (\mathcal{N} - \mathcal{N}')(cds, c^{-1}dr, du), \quad t \geq 0. \end{aligned}$$

Motivated by the above two identities, we define the Wiener process $\bar{B}_t := \frac{1}{\sqrt{c}} B_{ct}, t \geq 0$, and the independent Poisson random measures $\bar{\mathcal{N}}$ on $(0, \infty) \times (0, \infty) \times [-1, 1]$ by $\bar{\mathcal{N}}(ds, dr, du) := \mathcal{N}(cds, c^{-1}dr, du)$. It follows directly from the definition of a Poisson random measure that $\bar{\mathcal{N}}$ is a Poisson random measure with the same intensity measure as \mathcal{N} . With these definitions, the above calculation leads to

$$\begin{aligned} \bar{Z}_t^z &= \frac{z}{c} + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du) \right) \int_0^t \text{sign}(\bar{Z}_s^z) ds + \sigma \int_0^t \sqrt{|\bar{Z}_s^z|} d\bar{B}_s \\ &\quad + \int_0^t \int_0^{\frac{1}{|\bar{Z}_{s-}^z|}} \int_{-1}^1 \bar{Z}_{s-}^z (u-1) (\bar{\mathcal{N}} - \bar{\mathcal{N}}')(ds, dr, du), \quad t \geq 0. \end{aligned}$$

Hence, $Z^{z/c}$ and \bar{Z}^z satisfies the SDE (1.7) with initial condition z/c and both do not spend time at zero and satisfies the symmetry condition (2.21). But then the identity of one-dimensional marginals holds due to Step 1e) of the proof of Proposition 2.11. Finally, the Markov property proved in Proposition 2.11 implies the identification of the finite-dimensional marginals and the self-similarity is proved.

The statement about the Lamperti-Kiu quintuple is a direct consequence of the construction of Z^z via the SDE (1.7) and the definition of Lamperti-Kiu quintuples. \square

2.3. Proof of Theorem 1.4. Let us start with a simple reformulation of the Condition (1.4) in a restrictive setting:

Lemma 2.13. *Suppose $(P^z)_{z \geq 0}$ is a positive self-similar Markov process that only jumps towards the origin. Then there is a unique self-similar extension $(P^z)_{z \geq 0}$ of $(P^z)_{z > 0}$ under which the canonical process leaves zero continuously precisely if $\Psi(1) > 0$.*

Proof. The spectrally negative assumption implies (for instance by the Lévy-Khintchin representation) that $\Psi(\lambda) = \log E[e^{\lambda \xi_1}] < \infty$ for all $\lambda \geq 0$. Note furthermore that, if well-defined, the Laplace exponent $\lambda \mapsto \Psi(\lambda)$ is a convex function on $\mathbb{R}_{\geq 0}$.

First, suppose ξ drifts to $-\infty$, so that the existence of the claimed extension is equivalent to (1.4). Since the right-derivative $\Psi'(0)$ equals $E[\xi_1] < 0$, there is some $s > 0$ such that $\Psi(s) < 0$. Thus, the convexity of $\lambda \mapsto \Psi(\lambda)$ implies that (1.4) is equivalent to $\Psi(1) > 0$.

Next, suppose ξ does not drift to $-\infty$. Then the existence of the claimed extension is equivalent to (1.5) which is trivially fulfilled since all overshoots are zero because all jumps are negative. At the same time the convexity of Ψ and $\Psi'(0) = E[\xi_1] \geq 0$ imply $\Psi(\lambda) > 0$ for any $\lambda > 0$. Hence, the claimed equivalence is trivial in this latter case. \square

To find a necessary condition for \mathbb{R}_* -valued self-similar Markov processes to have an extension that leaves zero continuously we want to apply Condition (1.4) for a suitably derived positive self-similar Markov process. Since we are only interested in symmetric self-similar processes the suitable choice is the absolute value.

Lemma 2.14. *Suppose $(P^z)_{z \in \mathbb{R}}$ is a symmetric \mathbb{R}_* -valued self-similar Markov process with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) that satisfies Assumption (A) and define $(|P|^z)_{z \geq 0}$ as the law of $|Z|$ under $(P^z)_{z \in \mathbb{R}}$.*

- (a) $(|P|^z)_{z \geq 0}$ is a positive self-similar Markov process.
- (b) The Lamperti-transformed Lévy process $\xi^{|P|}$ of $(|P|^z)_{z \geq 0}$ satisfies

$$\Psi^{|P|}(1) = \Psi(1) + \int_{-1}^0 (|u| - 1)V(du),$$

where Ψ is the Laplace exponent of the Lévy process ξ with triplet (a, σ^2, Π) killed at rate q .

Proof. (a) The Markov property for $(|P|^z)_{z \geq 0}$ is inherited from $(P^z)_{z \in \mathbb{R}}$ due to the symmetry assumption. The self-similarity carries over trivially.

(b) To determine $\Psi^{|P|}(1)$ we use Proposition 2.1 twice. First recall from Proposition 1.2 that P^z can be expressed as

$$\begin{aligned} Z_t = z + & \left(\Psi(1) + \int_{-1}^0 (u - 1)V(du) \right) \int_0^t \text{sign}(Z_s) ds + \sigma \int_0^t \sqrt{|Z_s|} dB_s \\ & + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^1 Z_{s-}(u - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0. \end{aligned}$$

Taking absolute values as in the proof of Corollary 2.9 we find that $|P|^z$ is given by

$$\begin{aligned} |Z_t| &= \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) t + \sigma \int_0^t \sqrt{|Z_s|} d\bar{B}_s \\ &\quad + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^1 |Z_{s-}|(|u| - 1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0, \end{aligned}$$

with the Brownian motion $\bar{B}_t := \int_0^t \text{sign}(Z_s) dB_s$. Equivalently, we can write

$$\begin{aligned} |Z_t| &= \left(\Psi(1) + \int_{-1}^0 (|u| - 1)V(du) \right) t + \sigma \int_0^t \sqrt{|Z_s|} d\bar{B}_s \\ &\quad + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_0^1 |Z_{s-}|(u - 1)(\bar{\mathcal{N}} - \bar{\mathcal{N}}')(ds, dr, du), \quad t \leq T_0, \end{aligned} \tag{2.27}$$

where $\bar{\mathcal{N}}$ has intensity $ds \otimes dr \otimes (\bar{\Pi}(du) + V(-du))$ on $(0, \infty) \times (0, \infty) \times (0, 1)$. Comparing with (2.2) we can read off the Lévy triplet for $\xi^{|P|}$ and in particular the Laplace exponent evaluated at 1. \square

We can now finish the proof of our main result.

Proof of Theorem 1.4. Recall from Proposition 1.2 that the \mathbb{R}_* -valued self-similar symmetric Markov families obtained from real-valued self-similar Markov families by absorption at zero are completely characterized by Lamperti-Kiu quintuples (a, σ^2, Π, q, V) .

To see that Condition (1.9) is necessary we apply Lemmas 2.13 and 2.14: Suppose $(P^z)_{z \in \mathbb{R}}$ is a real-valued self-similar Markov process that leaves zero continuously. Then the Markov family $(|P^\dagger|^z)_{z \geq 0}$ obtained by absorption at zero is a positive self-similar Markov family with a self-similar extension that leaves zero continuously. The Laplace exponent of the Lamperti transformed Lévy process satisfies

$$\Psi^{|P^\dagger|}(1) = \Psi(1) + \int_{-1}^0 (|u| - 1)V(du)$$

which, as we showed in Lemma 2.13 has to be strictly positive.

Conversely, if Condition (1.9) is satisfied for a given quintuple (a, σ^2, Π, q, V) , then by Proposition 2.12, we constructed in Section 2.2 a real-valued self-similar Markov process with Lamperti-Kiu triplet (a, σ^2, Π, q, V) . Furthermore, the solutions Z^z leave zero continuously since the integrand of the Poissonian integral is zero at zero. \square

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